Fermionic partial transpose and non-local order parameters for SPT phases of fermions

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Main Ref:

KS-Shapourian-Gomi-Ryu, arXiv:1710.01886

Related refs:

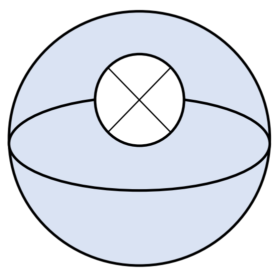
Shapourian-KS-Ryu, arXiv:1607.03896
KS-Ryu, arXiv:1607.06504
KS-Shapourian-Ryu, arXiv:1609.05970
Shapourian-KS-Ryu, 1611.07536

Take-home message

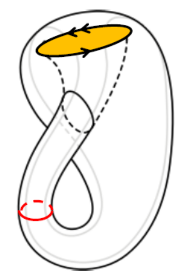
- Transpose $\sim$ time-reversal (TR) transformation for the imaginary time path-integral.

$$(\mathcal{O}_n \dots \mathcal{O}_2 \mathcal{O}_1)^{tr} = \mathcal{O}_1^{tr} \mathcal{O}_2^{tr} \cdots \mathcal{O}_n^{tr}$$

- The partial transpose enable us to simulate the partition function over some non-orientable manifolds in the operator formalism.



Real projective plane



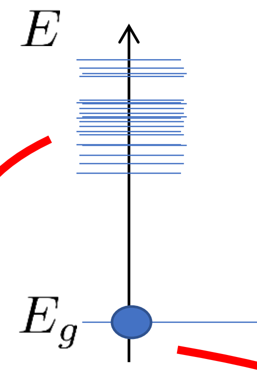
Klein bottle

- We proposed a new partial transpose for fermions such that it simulates the partition function over pin manifolds.

$$A^{tr_1} = \sum_{k_1, k_2, k_1+k_2 \in \text{even}} A_{p_1 \cdots p_{k_1}, q_1 \cdots q_{k_2}} i^{k_1} \underbrace{a_{p_1} \cdots a_{p_{k_1}}}_{I_1} \underbrace{b_{q_1} \cdots b_{q_{k_2}}}_{I_2}$$

Motivation

- How to detect symmetry protected topological (SPT) phases ($\sim$ gapped phases without ground state degeneracy)



- The classification of symmetry-protected topological phases
$\sim$ the classification of U(1)-valued topological partition functions
[Kapustin, Freed-Hopkins, $\cdots$]

$$Z[M] = \big|Z[M]\big| \times e^{iS_{\mathrm{top}}[M]}$$

Euclidian spacetime manifold

Excited states

Characterizes SPT phases

Motivation

- In SPT phases with time-reversal (TR) symmetry, the spacetime manifold *M* to detect nontrivial SPT phases is sometimes non-orientable.

  ✓ Ex: Haldane chain phase, topological insulator/superconductor,$\cdots$

- The partition function over a suitable non-orientable manifold is the "order parameter" of SPT phases.

- Ex: (1+1)d bosonic SPT phases with TR sym.

  ✓ "Order parameter" = real projective plane RP^2

$$e^{iS_{\mathrm{top}}(M)} = e^{i\pi\nu \int_M w_2(TM)}, \quad (\nu = 0, 1)$$

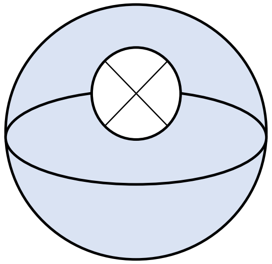
$$\Rightarrow e^{iS_{\mathrm{top}}(RP^2)} = (-1)^\nu.$$

  ✓ If the partition function over RP^2 is negative, the theory is in nontrivial SPT phases.

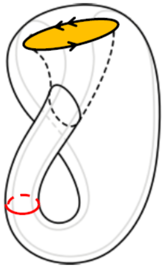
Motivation

- Non-orientable manifold???
- In cond-mat, we have only
  - ✓ Hamiltonian H or the ground state $|\psi\rangle$, and
  - ✓ TR operator T.
- How to make non-orientable manifold from a set of a ground state wave function and a TR operator?

$$|GS\rangle \,,\ T \quad \Longrightarrow$$



Real projective plane



Klein bottle

- An answer is to use the partial transpose.

TRS $\leftrightarrow$ Transpose

TRS $\leftrightarrow$ Transpose

- Given a TR operator, how to get the transpose?

- Let's consider the expectation value of the TR operator.

$$\langle \psi | T | \psi \rangle$$

- This is ill-defined, because T is anti-linear.

- However, the amplitude is well-defined.

- Some calculation:

$$
\begin{aligned}
|\langle\psi|T|\psi\rangle|^2 &= \langle\psi| U |\psi\rangle^* \langle\psi|^* U^\dagger |\psi\rangle \\
&= \mathrm{tr}[|\psi\rangle\langle\psi| U |\psi\rangle^* \langle\psi|^* U^\dagger] \\
&= \mathrm{tr}[\rho U \rho^* U^\dagger] \\
&= \mathrm{tr}[\rho U \rho^{tr} U^\dagger],
\end{aligned}
$$

Complex conjugate

Transpose in the operator algebra

- Here, $\rho = |\psi\rangle\langle\psi|$ and U is the unitary part of the TR operator, i.e. $T = UK$ with K the complex conjugate.

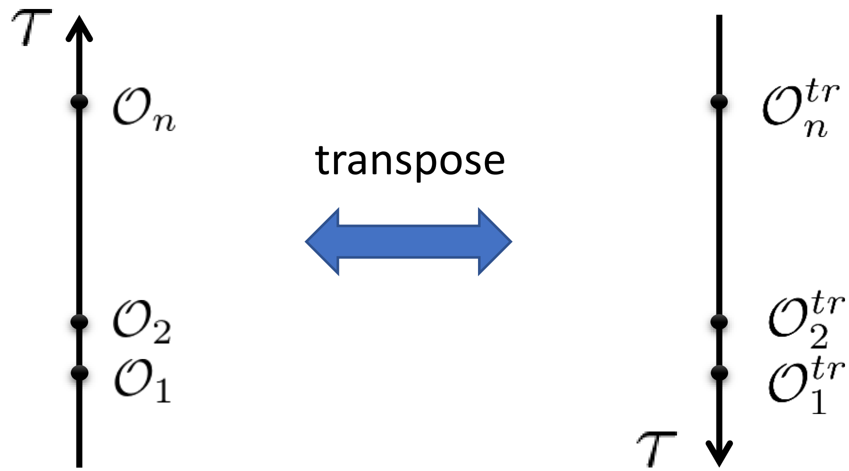
- We used the Hermiticity $\rho^\dagger = \rho$.

- In this way, a TR operator T induces a sort of the transpose in the operator algebra.

$$T = UK \quad \Rightarrow \quad U\rho^{tr}U^\dagger$$

- The transpose is understood as the time-reversal transformation in the imaginary time path-integral.

$$(\mathcal{O}_n \dots \mathcal{O}_2 \mathcal{O}_1)^{tr} = \mathcal{O}_1^{tr} \mathcal{O}_2^{tr} \cdots \mathcal{O}_n^{tr}$$



- Therefore, it is expected that the transpose can be used to "simulate" non-orientable manifolds.

- Advantage: the transpose operation is linear, so it can be applied to a subsystem of the real space.

$\Rightarrow$ Partial transpose

Partial transpose
and
non-orientable manifolds
in
bosonic systems

Bosonic transpose

- In bosonic (spin) systems, the operator algebra is the matrix algebra.
- The transpose is the matric transpose

$$\left(|i\rangle \langle j| \right)^{tr} = |j\rangle \langle i| \,.$$

- Given operator

$$A = \sum_{i,j} A_{i,j} |i\rangle \langle j| \,,$$

the transposed operator is given by

$$A^{tr} = \sum_{i,j} A_{i,j} |{\color{red} j}\rangle \langle {\color{red} i}| \,.$$

Bosonic partial transpose

- Divide the Hilbert space into two parts.

- Given a operator:

$$A = \sum_{ij,kl} A_{ij,kl} \left| i \in I_1, j \in I_2 \right\rangle \left\langle k \in I_1, l \in I_2 \right|$$

- The partial transpose on the subsystem I_1 is defined as the matrix transpose on I_1.

$$A^{tr_1} = \sum_{ij,kl} A_{ij,kl} \left| k \in I_1, j \in I_2 \right\rangle \left\langle i \in I_1, l \in I_2 \right|$$

An application: the non-local order parameter for the Haldane chain phase

- A model Hamiltonian: (1+1)d antiferromagnetic Heisenberg model

$$H = \sum_j \boldsymbol{S}_j \cdot \boldsymbol{S}_{j+1}.$$

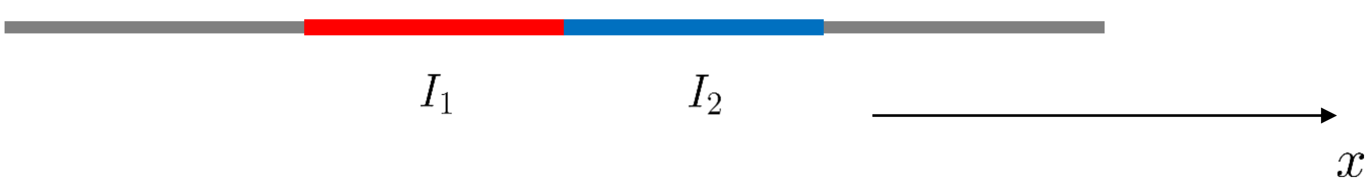
- TR symmetry $T = \bigotimes_j e^{i\pi S_j^y} K$

- The classification of SPT phases is known to be Z_2.

- For the S=1 spin system, the ground state is nontrivial SPT phase. (the Haldane chain).

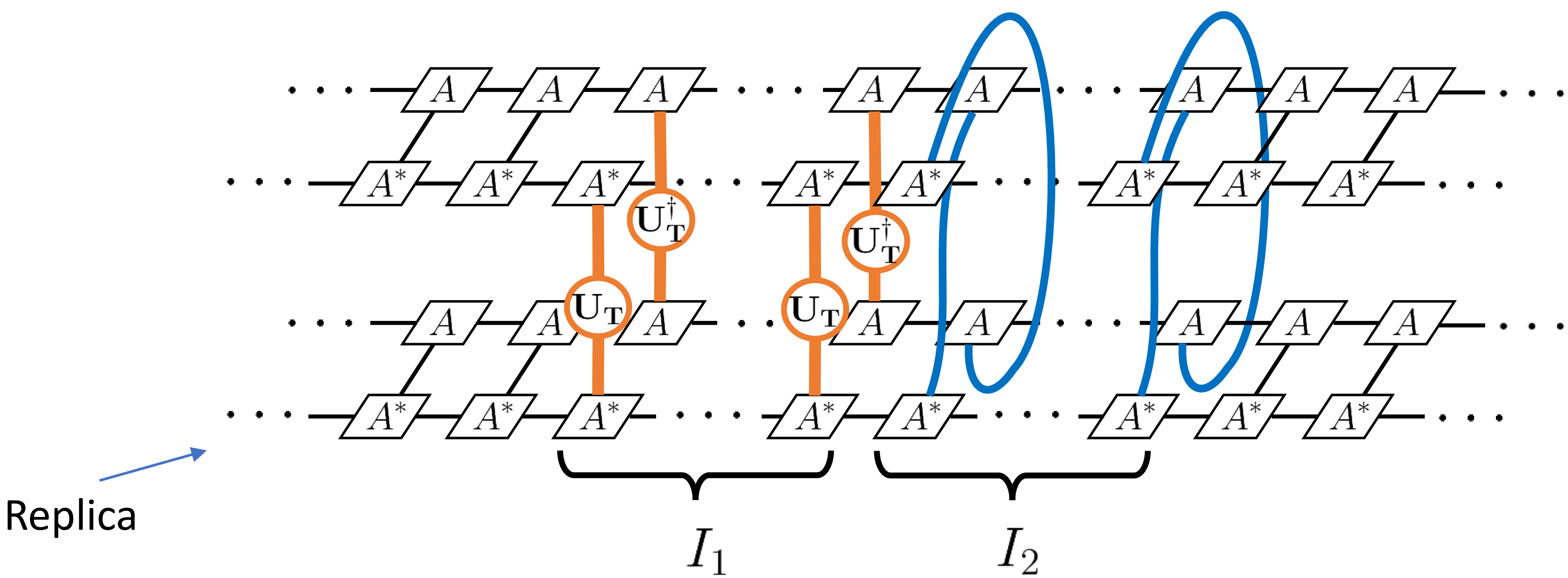
- The Z_2 "order parameter" is the partition function on RP^2 (real projective plane).

- Let's construct the Z_2 "order parameter" in the operator formalism.
- The rule of this game is:
  - ✓ Input data
    - Pure state (ground state) $|\psi\rangle$
    - TR operator $T = \bigotimes_j e^{i\pi S_j^y} K$
  - ✓ Out put = Z_2 order parameter
- Pollmann and Turner discovered that the Z_2 order parameter is the "partial transpose" on two adjacent intervals. [Pollmann-Turner '12]



- The partial transpose on two adjacent intervals. [Pollmann-Turner '12]

$$Z_{PT} := \mathrm{tr}\left[\rho_{I_1\cup I_2}\left(\prod_{j\in I_1} e^{i\pi S_j^y}\right)\left(\rho_{I_1\cup I_2}\right)^{tr_1}\left(\prod_{j\in I_1} e^{-i\pi S_j^y}\right)\right]$$

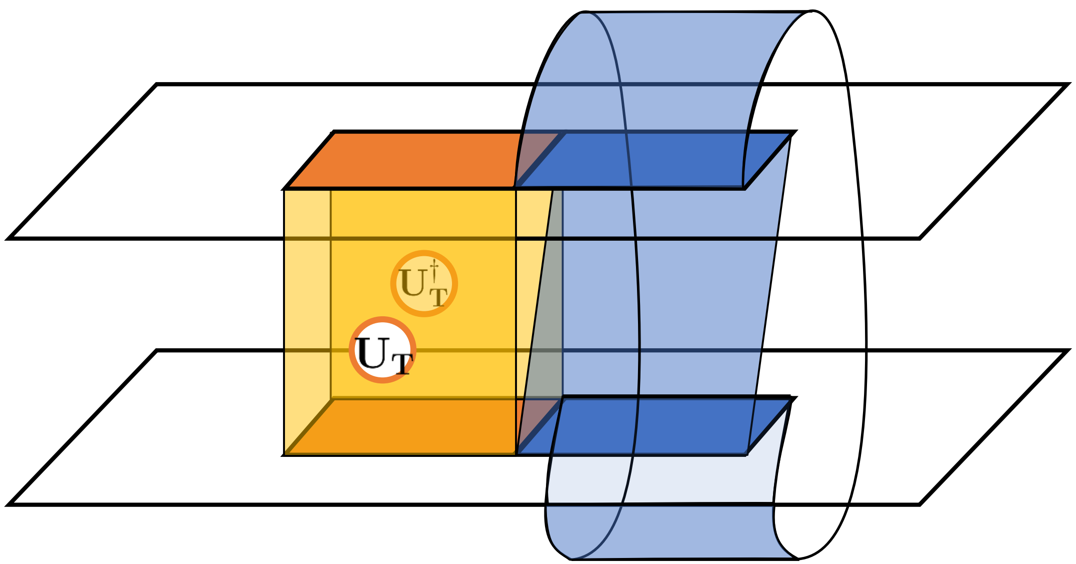


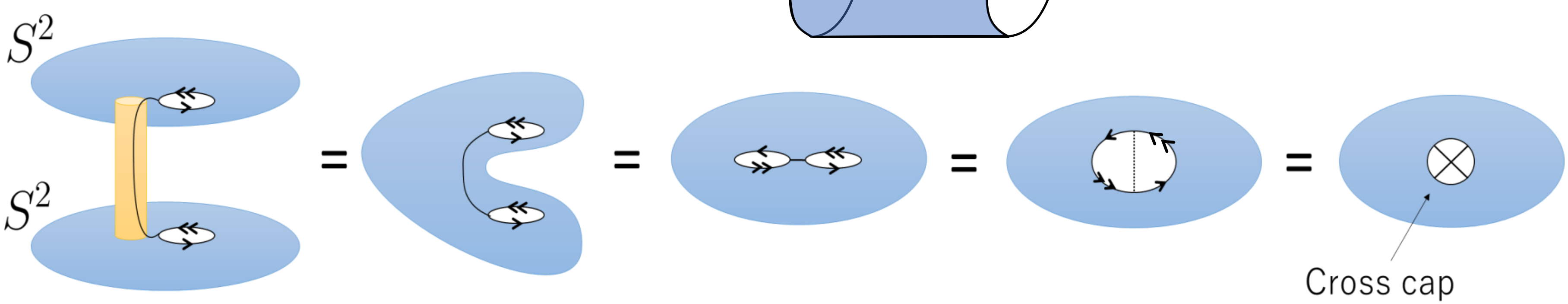
- Using the MPS, one can prove that the U(1) phase of Z_{PT} is quantized if intervals are large enough compared to the correlation length.

$$Z_{PT}/|Z_{PT}| \to \pm 1, \qquad |I_1|, |I_2| \gg \xi.$$

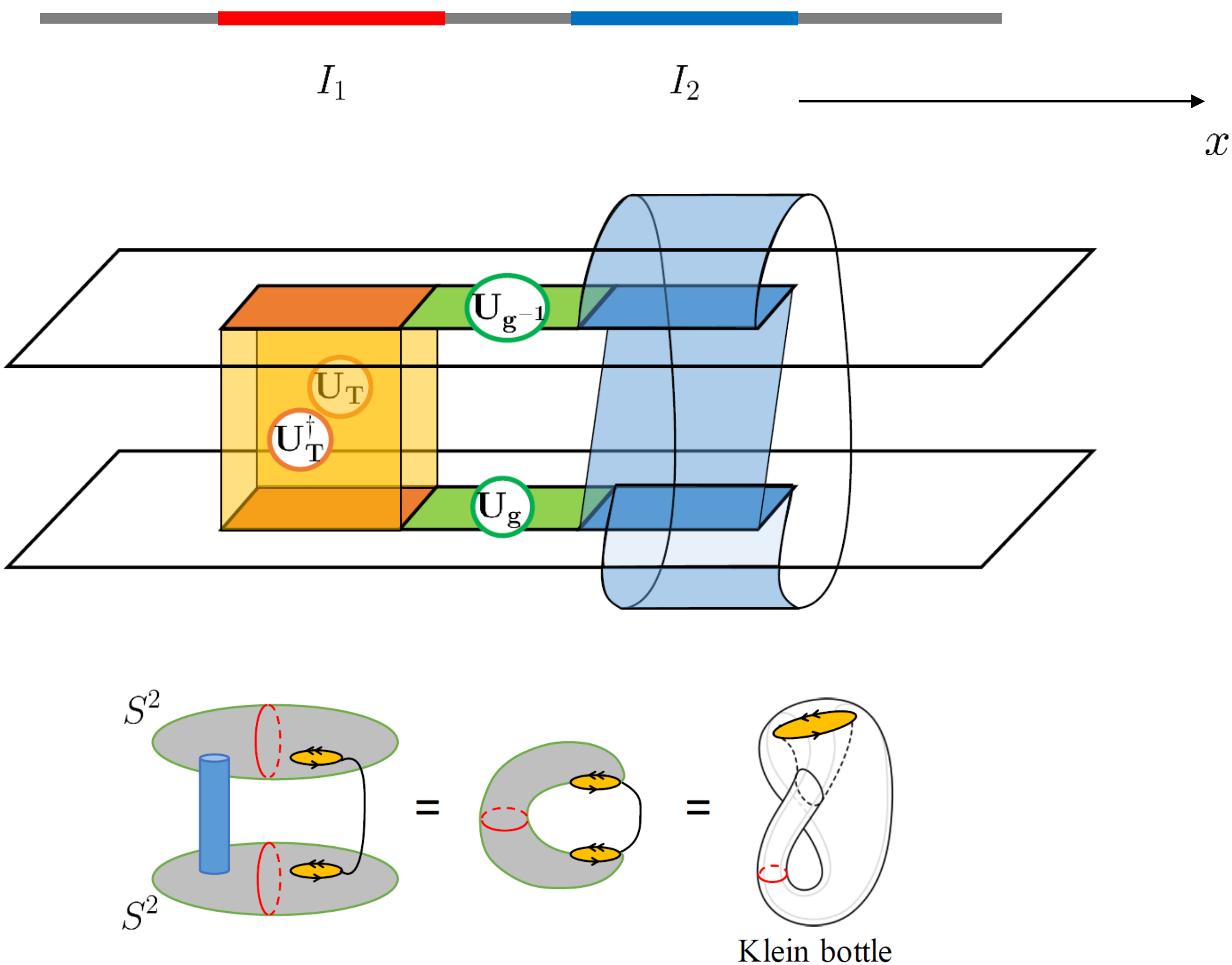
- The Pollmann-Turner invariant Z_{PT} is topologically equivalent to the partition function over RP². [KS-Ryu, '16]

$$Z := \mathrm{tr}\left[\rho_{I_1\cup I_2}\left(\prod_{j\in I_1} e^{i\pi S_j^y}\right)(\rho_{I_1\cup I_2})^{tr_1}\left(\prod_{j\in I_1} e^{-i\pi S_j^y}\right)\right]$$





- In the same way, the partial transpose for disjoint two intervals is topologically equivalent to the Klein bottle partition function. [Calabrese-Cardy-Tonni '12]



Partial transpose
and
non-orientable manifolds
in
fermionic systems

Transpose in the fermionic operator algebra

- Every operator can be expanded by Majorana fermions

$$A = \sum_{k=1}^{2N} \sum_{p_1 < p_2 \cdots < p_k} A_{p_1 \cdots p_k} c_{p_1} \cdots c_{p_k},$$

$$c_j^\dagger = c_j, \quad \{c_j, c_k\} = 2\delta_{jk}.$$

- Operator algebra = the algebra generated by the Majorana fermions (Clifford algebra).

- The transpose (linear anti-automorphism) would be defined as reversing the order of Majorana fermions.

$$(c_{p_1} c_{p_2} \cdots c_{p_k})^{tr} := c_{p_k} \cdots c_{p_2} c_{p_1}$$

$$(\alpha A + \beta B)^{tr} = \alpha A^{tr} + \beta B^{tr}, \quad (AB)^{tr} = B^{tr} A^{tr}.$$

- This transpose is “canonical” in the sense that the transpose is compatible with the basis change

$$V c_j V^\dagger = [\mathcal{O}_V]_{jk} c_k, \quad \mathcal{O}_V \in O(2N).$$

$$\begin{tikzcd} A \arrow[r, "V"] \arrow[d, "tr"'] \arrow[dr, phantom, "\circlearrowleft"] & VAV^\dagger \arrow[d, "tr"'] \\ A^{tr} \arrow[r, "V"] & VA^{tr}V^\dagger \end{tikzcd}$$

- We got the full transpose for fermions.

- A remark: we did not use a TR transformation to define the transpose of fermions. This can be compared with the transpose of bosons where there is no canonical transpose in the absence of a TR transformation.

Fermionic transpose and Grassmannian

- Does the fermionic transpose give the TR transformation in the imaginary time path-integral?

- Yes. Let us consider a simple TR transformation for complex fermions f_j as

$$T f_j^\dagger T^{-1} = f_j^\dagger, \qquad T\left|vac\right\rangle = \left|vac\right\rangle .$$

- The unitary part of the TR transformation T is found to be the particle-hole transformation

$$C_T f_j^\dagger C_T^{-1} = f_j, \qquad C_T\left|vac\right\rangle = \left|full\right\rangle .$$

- The transpose operation corresponding to the TR transformation T is

$$A \mapsto C_T A^{tr} C_T^\dagger$$

- For coherent state basis

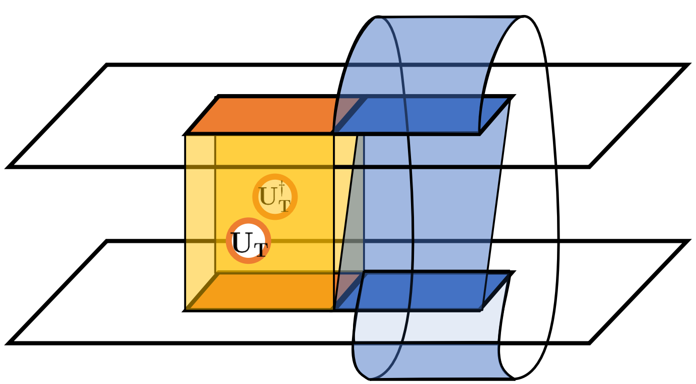
$$|\{\xi_j\}\rangle = e^{-\sum_j \xi_j f_j^\dagger} |vac\rangle \,,$$

the transpose with the particle-hole transformation reads

$$C_T \Big(|\{\xi_j\}\rangle \langle\{\chi_j\}| \Big)^{tr} C_T^\dagger = |\{i\chi_j\}\rangle \langle\{i\xi_j\}| \,.$$

- This is the desired TR transformation for the path-integral.

- Therefore, the partial transpose in fermions is expected to be used to simulate the real projective plane and the Klein bottle, as in the cases of bosons.



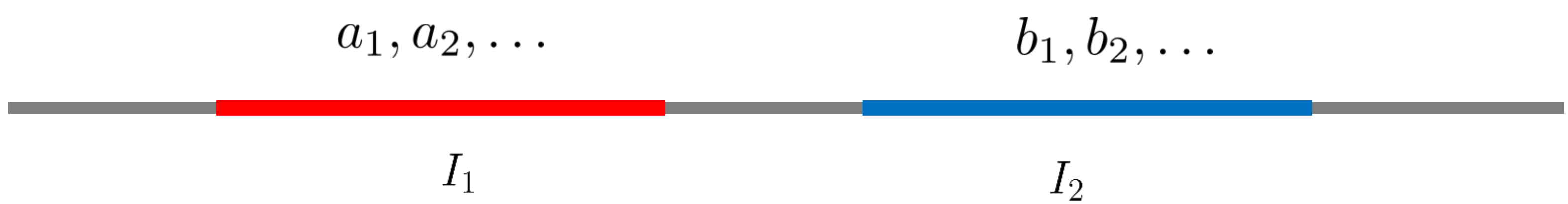
= RP^2

Fermionic partial transpose

[KS-Shapourian-Gomi-Ryu, 1710.01886,
cf. Shapourian-KS-Ryu, 1607.03896;
Shapourian-Ryu, 1804.08637]

- What is the partial transpose for fermions?

- Introduce two subsystems of degrees of freedom.



$$A = \sum_{k_1, k_2} A_{p_1 \cdots p_{k_1}, q_1 \cdots q_{k_2}} \underbrace{a_{p_1} \cdots a_{p_{k_1}}}_{I_1} \underbrace{b_{q_1} \cdots b_{q_{k_2}}}_{I_2}$$

- We want to define the partial transpose "A^{tr_1}“ on the subsystem I_1.

- It is natural to impose the following three good properties on the partial transpose :

1. Preserve the identity:

$$(\mathrm{Id})^{tr_1} = \mathrm{Id}$$

2. The successive partial transposes on I_1 and I_2 goes back to the full transpose:

$$(A^{tr_1})^{tr_2} = A^{tr}$$

3. The partial transpose is compatible with the basis changes preserving subsysems $I_1 \cup I_2$.

$$(VAV^\dagger)^{tr_1} = VA^{tr_1}V^\dagger,$$

$$Va_jV^\dagger = [\mathcal{O}_{I_1}]_{jk}a_k, \quad Vb_jV^\dagger = [\mathcal{O}_{I_2}]_{jk}b_k.$$

- For generic operators there is no solution that meets the above three conditions.

- For operators preserving the total fermion parity (like the density matrix)

$$A = \sum_{k_1, k_2, k_1+k_2 \in \text{even}} A_{p_1 \cdots p_{k_1}, q_1 \cdots q_{k_2}} \underbrace{a_{p_1} \cdots a_{p_{k_1}}}_{I_1} \underbrace{b_{q_1} \cdots b_{q_{k_2}}}_{I_2},$$

  there is the unique solution.

- The partial transpose for the subsystem I_1 is the scalar multiplication by i^{k_1} which depends on the number of the Majorana fermions in the subspace I_1.

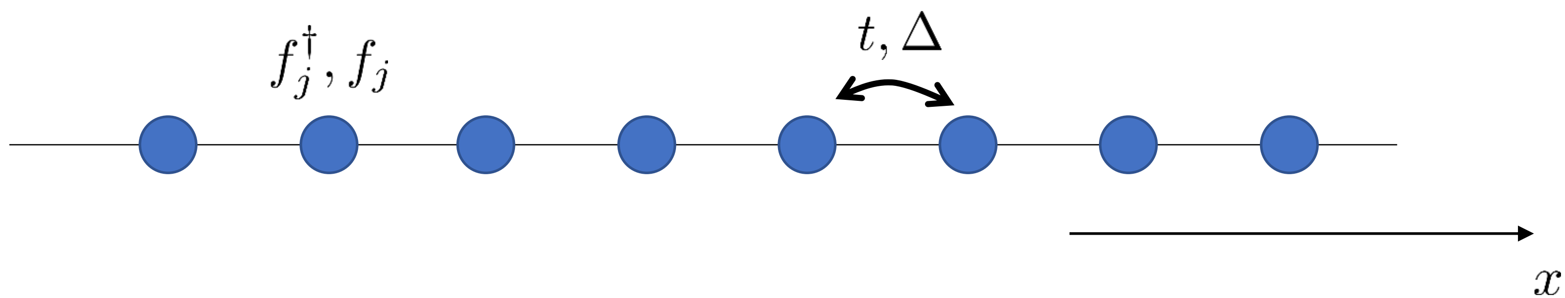
$$\boxed{A^{tr_1} = \sum_{k_1, k_2, k_1+k_2 \in \text{even}} A_{p_1 \cdots p_{k_1}, q_1 \cdots q_{k_2}} i^{k_1} \underbrace{a_{p_1} \cdots a_{p_{k_1}}}_{I_1} \underbrace{b_{q_1} \cdots b_{q_{k_2}}}_{I_2}}$$

Shapourian-KS-Ryu, 1607.03896.; KS-Shapourian-Gomi-Ryu, 1710.01886.

An application: the non-local order parameter for 1d superconductors with TR symmetry

- A model Hamiltonian: (1+1)d p-wave superconductor (Kitaev chain)

$$H = \sum_j \left[-t f_{j+1}^\dagger f_j + \Delta f_{j+1}^\dagger f_j^\dagger + h.c. \right] - \mu \sum_j f_j^\dagger f_j$$



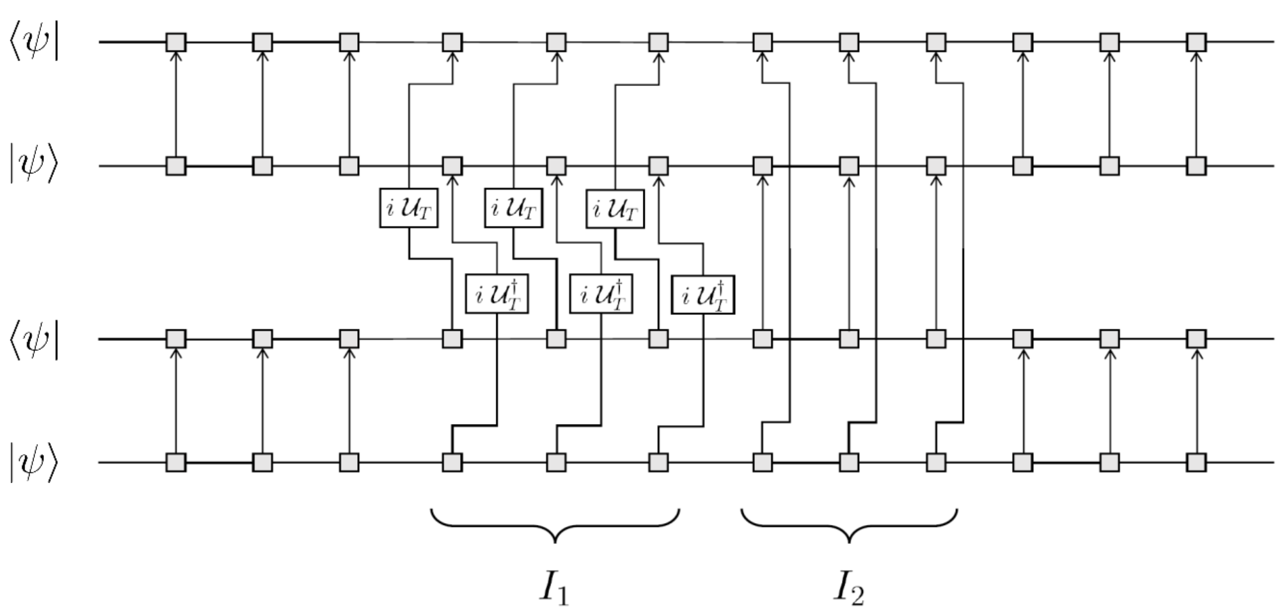
- TR symmetry $\quad T f_j^\dagger T^{-1} = f_j^\dagger, \qquad T\left|vac\right\rangle = \left|vac\right\rangle.$

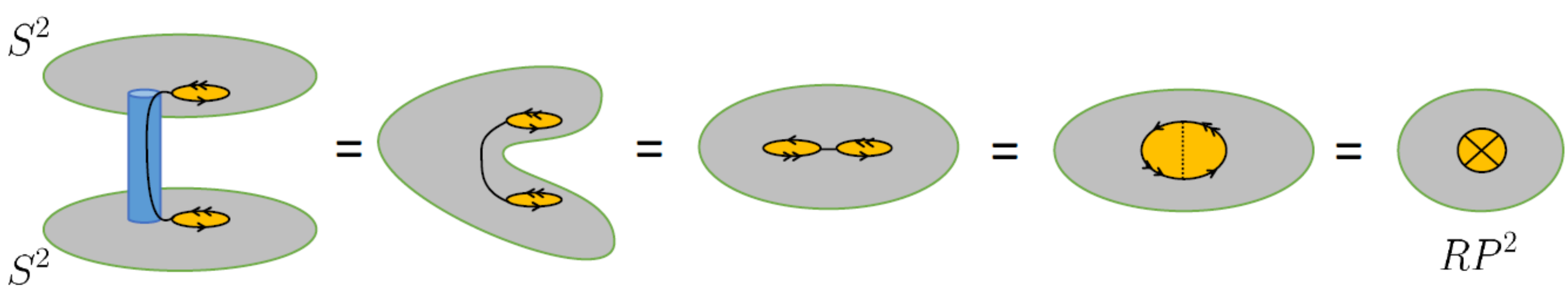
- The classification of SPT phases is known to be Z_8. [Fidkowski-Kitaev '10]

- The Z_8 "order parameter" is the partition function on RP² (real projective plane). $\quad \Omega_2^{Pin_-}(pt) = \mathbb{Z}_8$

- The Z_8 order parameter in operator formalism is the partial transpose on adjacent two intervals

$$Z = \mathrm{tr}_{I_1 \cup I_2}\left[\rho_{I_1\cup I_2} C_T^{I_1} \rho_{I_1\cup I_2}^{tr_1} [C_T^{I_1}]^\dagger\right]$$

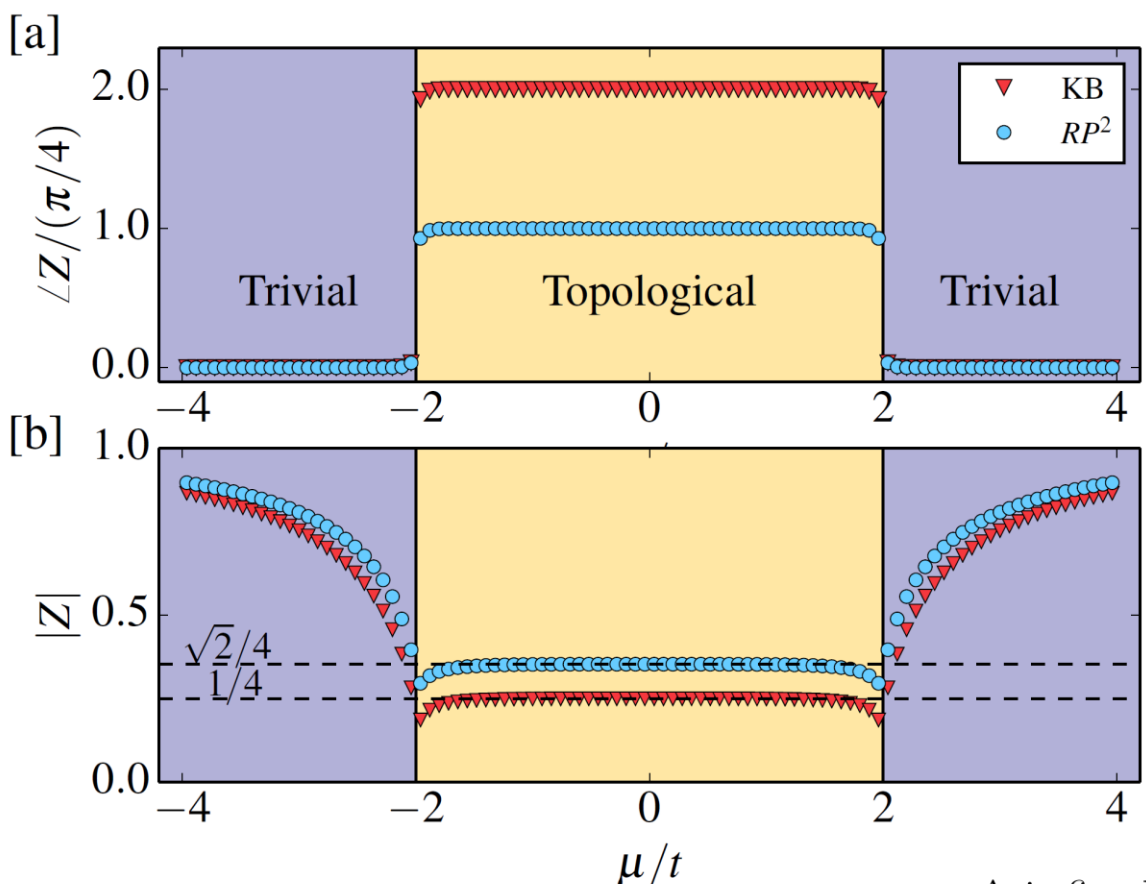




- Numerical calculation [arXiv:1607.03896]

$$H = -t\sum_j \left[f_j^\dagger f_j - \Delta f_{j+1}^\dagger f_j^\dagger + h.c. \right] - \mu \sum_j f_j^\dagger f_j$$

$$Z = \mathrm{tr}_{I_1 \cup I_2}\left[\rho_{I_1\cup I_2} C_T^{I_1} \rho_{I_1\cup I_2}^{tr_1} [C_T^{I_1}]^\dagger \right]$$



Δ is fixed to be $\Delta = t$.

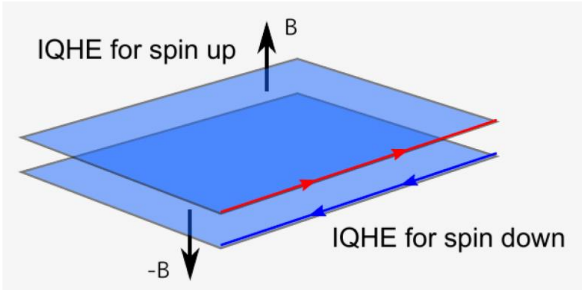
Manybody Z_2 Kane-Mele invariant

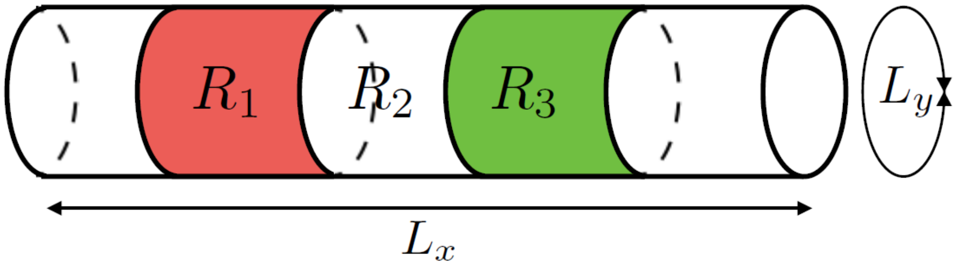
- (2+1)d topological insulator (TR symmetry with Kramers)

- The generating manifold is the Klein bottle × S^1 with a unit magnetic flux. [Witten '16]

- Combine two technics:

  ✓ Disjoint partial transpose -> Klein bottle
  ✓ Twist operator -> a unit magnetic flux





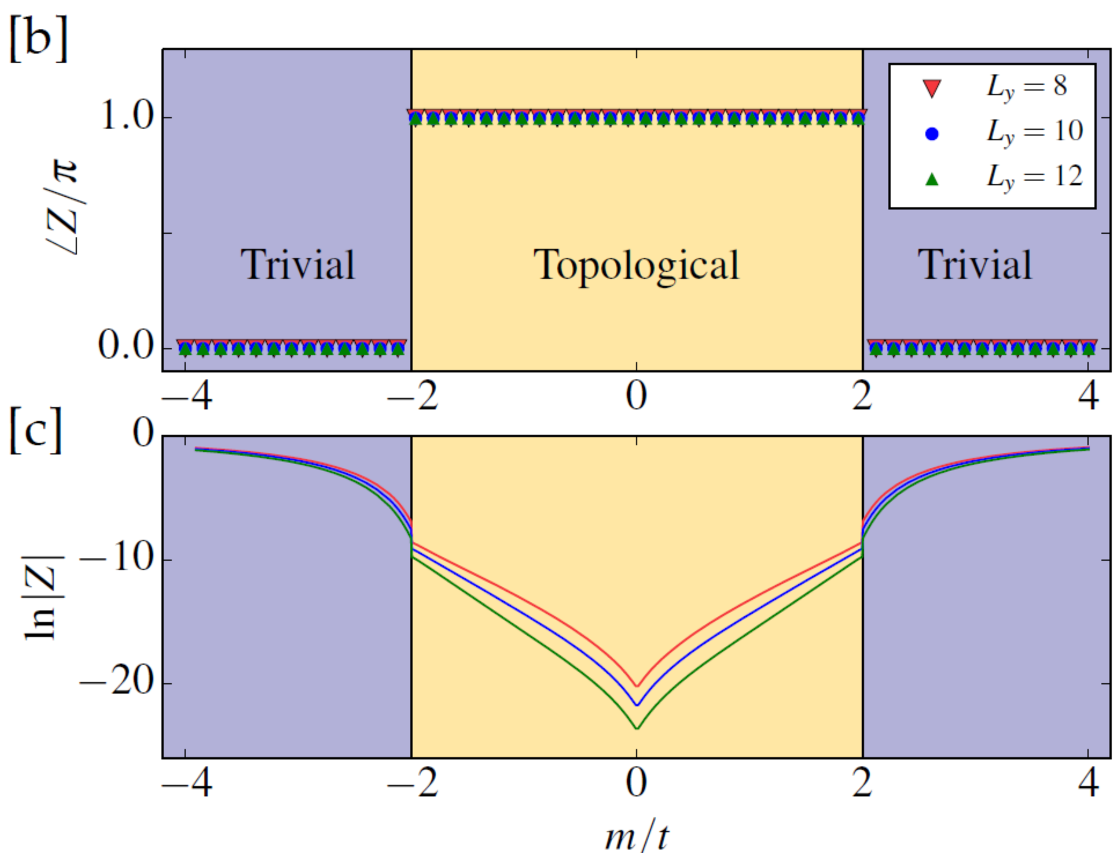
$$\rho^{\pm}_{R_1\cup R_3} := \mathrm{Tr}_{\overline{R_1\cup R_3}}\left[e^{\pm\sum_{(x,y)\in R_2}\frac{2\pi i y}{L_y}\hat{n}(x,y)}\left|GS\right\rangle\left\langle GS\right|\right],$$

$$Z := \mathrm{Tr}_{R_1\cup R_3}\left[\rho^{+}_{R_1\cup R_3}C_T^{I_1}[\rho^{-}_{R_1\cup R_3}]^{tr_1}[C_T^{I_1}]^{\dagger}\right].$$

Manybody Z_2 Kane-Mele invariant

- Numerical calculation for a free fermion model

$$Z := \mathrm{Tr}_{R_1\cup R_3}\left[\rho^+_{R_1\cup R_3} C_T^{I_1} [\rho^-_{R_1\cup R_3}]^{tr_1} [C_T^{I_1}]^\dagger\right].$$

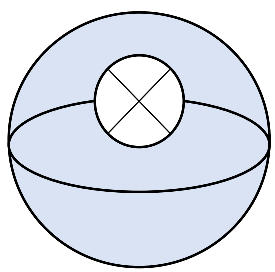


Take-home message

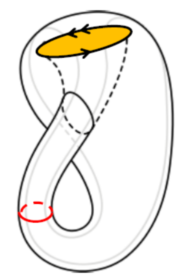
- Transpose $\sim$ time-reversal (TR) transformation for the imaginary time path-integral.

$$(\mathcal{O}_n \ldots \mathcal{O}_2 \mathcal{O}_1)^{tr} = \mathcal{O}_1^{tr} \mathcal{O}_2^{tr} \cdots \mathcal{O}_n^{tr}$$

- The partial transpose enable us to simulate the partition function over non-orientable manifolds in the operator formalism.



Real projective plane



Klein bottle

- Developed the partial transpose for fermions in the operator formalism

$$A^{tr_1} = \sum_{k_1, k_2, k_1+k_2 \in \mathrm{even}} A_{p_1 \cdots p_{k_1}, q_1 \cdots q_{k_2}} i^{k_1} \underbrace{a_{p_1} \cdots a_{p_{k_1}}}_{I_1} \underbrace{b_{q_1} \cdots b_{q_{k_2}}}_{I_2}$$