TBA equations and resurgent quantum mechanics

arXiv: 1811.04812 [hep-th], JHEP 01 (2019) 228
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June 14, 2019
E-Lab seminar, Nagoya University

Introduction

Introduction

We consider the second order differential equation:

$$-\hbar^2 \frac{d^2}{dx^2}\psi(x) + (V(x) - E)\psi(x) = 0.$$

where $V(x)$ is a polynomial in x.

This ODE appears in

- quantum mechanics: 1D Schrödinger equation
- 4d SUSY gauge theory
  quantum SW curve in $N = 2$ gauge theories
  minimal surface in AdS spacetime ($N = 4$ SYM)

The standard WKB method produces asymptotic expansions in $\hbar$ for the solutions to the Schrödinger equation.

exact WKB method [Voros 1981, Sato-Aoki-Kawai-Takei, ...]

- Borel resummations and Laplace transformation
- resurgence (perturbative $\longleftrightarrow$ non-perturbative)
- The exact WKB periods and the exact quantization condition determines the exact spectrum of QM.
- the WKB periods show discontinuity across the Stokes lines
- Voros' idea: The classical periods and their discontinuity structure determines the exact WKB periods.
  analytic bootstrap or the Riemann-Hilbert problem
- explicitly worked out for cubic and quartic potentials only

ODE/IM correspondence

The ODE/IM correspondence [Dorey-Tateo 1998]

- a relation between spectral analysis approach of ordinary differential equation (ODE), and the "functional relations" approach to 2d quantum integrable model (IM).
- Stokes coefficients of the solutions satisfy functional relations
- Baxter's T-Q relation
  T-system, Y-system
  $\Longrightarrow$ NLIE, Thermodynamic Bethe Ansatz (TBA) equations
- TBA equations solve the spectral determinants and the exact WKB periods of QM with the monic potential $V(x) = x^{2M}$.

KI-Mariño-Shu 2018

We present the TBA equations governing the exact WKB periods for general polynomial potential. These TBA also provide a generalization of the ODE/IM correspondence.

We will discuss

- the relation between the discontinuity formula in Quantum Mechanics and the TBA equations
- generalization of the ODE/IM correspondence

for arbitrary polynomial potential.

We then apply the TBA equations to solve the spectral problem in QM.

Introduction

Exact WKB and resurgent quantum mechanics

Generalized ODE/IM correspondence

Example: cubic potential

Conclusions and outlook

Exact WKB and resurgent quantum mechanics

Exact WKB method

the stationary Schrödinger equation for a non-relativistic particle in a potential $V(q)$ and with energy E:

$$-\hbar^2\psi''(q) + (V(q) - E)\psi(q) = 0.$$

the WKB solution:

$$\psi(q) = \exp\left[\frac{i}{\hbar}\int^q Q(q')dq'\right].$$

$$Q(q) = \sum_{k=0}^{\infty} Q_k(q)\hbar^k = P(q) + \frac{i\hbar}{2}\frac{d}{dq}\log P(q).$$

$$P(q) = \sum_{n\geq 0} p_n(q)\hbar^{2n},$$

$p_0(q) = p(q) := (E - V(q))^{1/2}$ and $p_n(q)$ are determined recursively.

WKB periods and Voros symbols

potential: polynomial in q

$$V(q) = q^{r+1} + u_1 q^r + \cdots + u_r q$$

WKB curve: hyperelliptic Riemann surface

$$y^2 = 2(E - V(q)).$$

WKB periods or *quantum periods*:

$$\Pi_\gamma(\hbar) = \oint_\gamma P(q)dq, \qquad \gamma \in H_1(\Sigma_{\mathrm{WKB}}).$$

$$\Pi_\gamma(\hbar) = \sum_{n\geq 0} \Pi_\gamma^{(n)} \hbar^{2n}, \qquad \Pi_\gamma^{(n)} = \oint_\gamma p_n(q)dq.$$

Voros multiplier or *Voros symbol*

$$\mathcal{V}_\gamma = \exp\left(\frac{i}{\hbar}\Pi_\gamma\right).$$

Borel resummation

The WKB expansion of the quantum periods:

$$\Pi_\gamma(\hbar) = \sum_{n\geq 0} \Pi_\gamma^{(n)} \hbar^{2n}, \qquad \Pi_\gamma^{(n)} = \oint_\gamma p_n(q) dq.$$

- asymptotic expansion: $\Pi_\gamma^{(n)} \sim (2n)!$.
- The Borel transformation of the WKB or quantum period:

$$\widehat{\Pi}_\gamma(\xi) = \sum_{n\geq 0} \frac{1}{(2n)!} \Pi_\gamma^{(n)} \xi^{2n},$$

- analytic in a neighborhood of the origin in the ξ-plane
- it can be analytically continued to a function on the complex plane, displaying in general various types of singularities

The Borel resummation of the quantum period (Laplace transformation)

$$s\left(\Pi_\gamma\right)(\hbar) = \frac{1}{\hbar}\int_0^\infty e^{-\xi/\hbar}\widehat{\Pi}_\gamma(\xi)d\xi, \qquad \hbar \in \mathbf{R}_{>0}.$$

Borel resummation along a direction in the complex plane, specified by an angle φ:

$$s_\varphi\left(\Pi_\gamma\right)(\hbar) = \frac{1}{\hbar}\int_0^{e^{i\varphi}\infty} e^{-\xi/\hbar}\widehat{\Pi}_\gamma(\xi)d\xi.$$

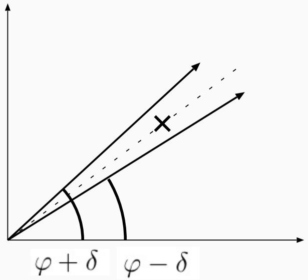
If this integral converges when $\hbar$ is small enough, the quantum period is said to be *Borel summable*.

- singularity in the φ-direction in the ξ-plane
- the *lateral Borel resummations* along a direction φ:

$$s_{\varphi\pm}(\Pi_\gamma)\left(e^{i\varphi}\hbar\right) = \lim_{\delta\to 0+} s\left(\Pi_\gamma\right)\left(e^{i\varphi\pm i\delta}\hbar\right), \qquad \hbar \in \mathbf{R}_{>0},$$

- The discontinuity across the direction φ

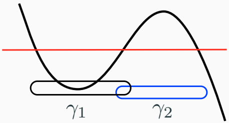
$$\mathrm{disc}_\varphi\left(\Pi_\gamma\right) = s_{\varphi+}(\Pi_\gamma) - s_{\varphi-}(\Pi_\gamma).$$



Delabaere-Pham formula

WKB periods

- cycle γ_1: a classically allowed interval $E > V(x)$
- cycle γ_2: a classically forbidden interval $E < V(x)$



- Π_{γ_1} not Borel summable
- Π_{γ_2} Borel summable

$$\mathrm{disc}\left(\Pi_{\gamma_1}\right)(\hbar) = -i\hbar \log\left(1 + \exp\left(-\frac{i}{\hbar}\Pi_{\gamma_2}(\hbar)\right)\right).$$

[Delabaere-Pham 1999, Iwaki-Nakanishi]

Exact quantization condition

Bohr-Sommerfeld quantization condition

$$s(\Pi_\gamma)(\hbar) = 2\pi\hbar\left(n + \frac{1}{2}\right), \quad n = 0, 1, 2, \dots$$

$\Longrightarrow E = E_n(\hbar)$: perturbative spectrum

Exact quantization condition [V, DP, Zinn-Justin,..., Grassi-Marino]

$$2\cos(\frac{1}{2\hbar}s(\Pi_p)(\hbar)) + e^{-\frac{1}{2\hbar}s(\Pi_{np})(\hbar)} = 0 \quad \text{(cubic potential)}$$

$\Longrightarrow E = E_n(\hbar)$: energy spectrum

$\frac{1}{\hbar} = x_n(E)$: Voros spectrum

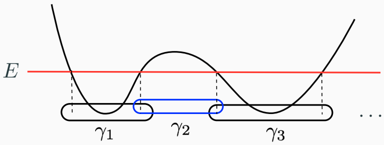
Generalized ODE/IM correspondence

TBA equations from DP formula

- $V_{r+1}(q)$: a polynomial potential of degree $r+1$
- all the turning points q_i, $i = 1, \cdots, r+1$, are real and different $(q_1 < q_2 < \cdots < q_{r+1}.)$
- γ_{2i-1}: $[q_{2i}, q_{2i}]$ classically allowed
  γ_{2i}: $[q_{2i}, q_{2i+1}]$ classically forbidden
- period integrals

$$m_{2i-1} = \Pi^{(0)}_{\gamma_{2i-1}} = 2\int_{q_{2i-1}}^{q_{2i}} p(q)dq, \; m_{2i} = i\,\Pi^{(0)}_{\gamma_{2i}} = 2i\int_{q_{2i}}^{q_{2i+1}} p(q)dq,$$

are real and positive.



TBA equations

- discontinuity of $\Pi_{\gamma_{2i-1}}$ is determined by the DP formula.

$$\operatorname{disc}\Pi_{\gamma_{2i-1}} = -i\hbar\log\left(1+\exp\left(-\frac{i}{\hbar}\Pi_{\gamma_{2i-2}}(\hbar)\right)\right) - i\hbar\log\left(1+\exp\left(-\frac{i}{\hbar}\Pi_{\gamma_{2i}}(\hbar)\right)\right),$$

- discontinuity of $\Pi_{\gamma_{2i}}$: $\hbar \to \pm i\hbar$ ($E - V(x) \to V(x) - E$)

Introduce the spectral parameter $\theta = -\log\hbar$ ($e^{\theta} = \frac{1}{\hbar}$)

$$-i\epsilon_{2i-1}\left(\theta + \frac{i\pi}{2} \pm i\delta\right) = \frac{1}{\hbar}s_{\pm}\left(\Pi_{\gamma_{2i-1}}\right)(\hbar), \quad -i\epsilon_{2i}(\theta) = \frac{1}{\hbar}s\left(\Pi_{\gamma_{2i}}\right)(\hbar),$$

The DP-formula reads

$$\operatorname{disc}_{\frac{\pi}{2}}\epsilon_a(\theta) = L_{a-1}(\theta) + L_{a+1}(\theta), \qquad a = 1,\cdots,r,$$

$$L_a(\theta) = \log\left(1 + e^{-\epsilon_a(\theta)}\right),$$

$L_0 = L_{r+1} = 0$

Discontinuity and integral equations

Integral transformation

$$F(\zeta) = \int_0^\infty \frac{dx}{x} \frac{x+\zeta}{x-\zeta} f(x) = \int_{-\infty}^\infty d\theta' \coth\frac{\theta-\theta'}{2} f(e^{\theta'})$$

$\zeta = e^\theta$, $x = e^{\theta'}$

The discontinuity of $F(\zeta)$ along the positive real axis

$$F(\zeta+i\epsilon) - F(\zeta-i\epsilon) = \int_0^\infty \frac{dx}{x} \frac{4i\epsilon x}{(x-\zeta)^2+\epsilon^2} f(x) \to 2\pi i f(\zeta)$$

asymptotics $\epsilon_a(\theta) \to m_a e^\theta$ $(\theta \to \infty)$
discontinuity ar $\pi/2$ and $-\pi/2 \Longrightarrow$ TBA equations

$$\epsilon_a(\theta) = m_a e^\theta - \int_{-\infty}^\infty \frac{L_{a-1}(\theta')}{\cosh(\theta-\theta')} \frac{d\theta'}{2\pi} - \int_\infty^\infty \frac{L_{a+1}(\theta')}{\cosh(\theta-\theta')} \frac{d\theta'}{2\pi},$$

WKB periods and pseudo energy

- $\epsilon_a(\theta)$: energy of a pseudo particle of the integrable model
- expansion in e^θ

$$\epsilon_a(\theta) \sim m_a e^\theta + \sum_{n\geq 1} m_a^{(n)} e^{(1-2n)\theta},$$

$$m_a^{(n)} = \frac{(-1)^n}{\pi} \int_{\mathbf{R}} e^{(2n-1)\theta} \left(L_{a-1}(\theta) + L_{a+1}(\theta)\right) d\theta.$$

- WKB periods

$$\Pi_\gamma^{(n)} = \mathcal{O}_n \Pi_\gamma^{(0)}$$

$\mathcal{O}_n$: differential operator with respect to moduli parameters

- $$m_{2i-1}^{(n)} = (-1)^n \Pi_{\gamma_{2i-1}}^{(n)}, \qquad m_{2i}^{(n)} = i\Pi_{\gamma_{2i}}^{(n)}.$$

We can check these relations numerically.

generalized ODE/IM correspondence

ODE

$$\left(-\partial_z^2 + z^{r+1} + \sum_{a=1}^{r} b_a z^{r-a}\right)\psi(z,b_a) = 0 \quad z \in \mathbf{C}$$

- invariant under the rotation (Symanzik rotation)

$$(z,b_a) \to (\omega z, \omega^{a+1} b_a), \qquad \omega = e^{\frac{2\pi i}{r+3}}.$$

- asymptotically decaying solution along the positive real axis

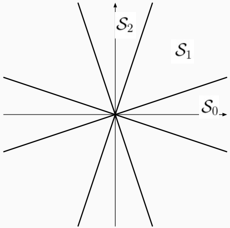
$$y(z,b_a) \sim \frac{1}{\sqrt{2i}} z^{n_r} \exp\Big(-\frac{2}{r+3} z^{\frac{r+3}{2}}\Big),$$

Stokes sector

$\mathcal{S}_k = \Big\{ z \in \mathbf{C} : \Big|\arg(z) - \frac{2k\pi}{r+3}\Big| < \frac{\pi}{r+3}\Big\}.$

decaying solutions in $\mathcal{S}_k$:

$$y_k(z,b_a) = \omega^{\frac{k}{2}} y(\omega^{-k} z, \omega^{-(a+1)k} b_a).$$



Y-functions and Wronskian

Wronskian

$$W_{k_1,k_2}(b_a) \equiv y_{k_1}(z,b_a)\partial_z y_{k_2}(z,b_a) - y_{k_2}(z,b_a)\partial_z y_{k_1}(z,b_a)$$

- independent of z, $W_{0,1}(=1)$.
- periodicity

$$W_{k_1+1,k_2+1}(b_a) = W^{[2]}_{k_1,k_2}(b_a).$$

$$f^{[j]}(z,b_a) := f(\omega^{-j/2}z, \omega^{-j(a+1)/2}b_a).$$

Y-function

$$\mathcal{Y}_{2j}(b_a) = \frac{W_{-j,j}(b_a)W_{-j-1,j+1}(b_a)}{W_{-j-1,-j}(b_a)W_{j,j+1}(b_a)},$$

$$\mathcal{Y}_{2j+1}(b_a) = \left[\frac{W_{-j-1,j}(b_a)W_{-j-2,j+1}(b_a)}{W_{-j-2,-j-1}(b_a)W_{j,j+1}(b_a)}\right]^{[+1]},$$

Y-system

Y-system ($\leftarrow$ Plücker identities for 2×2 determinants)

$$\mathcal{Y}_s^{[+1]}(b_a)\mathcal{Y}_s^{[-1]}(b_a) = \Big(1 + \mathcal{Y}_{s-1}(b_a)\Big)\Big(1 + \mathcal{Y}_{s+1}(b_a)\Big).$$

boundary conditions: $\mathcal{Y}_0 = \mathcal{Y}_{r+1} = 0$

A_r-type Y-system

- $b_1 = \cdots = b_{r-1} = 0$, $b_r = E$ (monomial pot.) [Dorey-Tateo]
- multiple spectral parameters $b_1, \cdots, b_r$
- It is difficult to write down TBA equations
  6-term TBA for cubic potential Masoero 1005.1046
  NLIE for x^6-potential Suzuki 0003066

generalized ODE/IM

Introduce scaled variables

$$q = \zeta^{\frac{2}{r+3}} z, \quad u_a = -\zeta^{\frac{2(a+1)}{r+3}} b_a, \qquad a = 1, 2, \cdots, r.$$

$$\left(-\zeta^2 \partial_q^2 + q^{r+1} - \sum_{a=1}^{r} u_a q^{r-a} \right) \hat{\psi}(x, u_a, \zeta) = 0.$$

- regard ζ as a new spectral parameter ($\zeta = \hbar$ in QM)
- Symanzik rotation $\leftrightarrow$ rotation of ζ

$$\hat{y}(q, u_a, \zeta) = y(z, b_a) = y(\zeta^{-\frac{2}{r+3}} q, -\zeta^{-\frac{2(a+1)}{r+3}} u_a).$$

$$(\omega^{-k} z, \omega^{-(a+1)k} b_a) = \left((e^{i\pi k}\zeta)^{-\frac{2}{r+3}} q, -(e^{i\pi k}\zeta)^{-\frac{2(a+1)}{r+3}} u_a \right).$$

- subdominant solution in the sector $\mathcal{S}_k$

$$\hat{y}_k(q, u_a, \zeta) = \omega^{\frac{k}{2}} \hat{y}(q, u_a, e^{i\pi k}\zeta).$$

Y-system

Wronskian

$$\hat{W}_{k_1,k_2}(\zeta,u_a)=\zeta^{\frac{2}{r+3}}\Big(\hat{y}_{k_1}\partial_q\hat{y}_{k_2}-\hat{y}_{k_2}\partial_q\hat{y}_{k_1}\Big)(q,u_a,\zeta)=W_{k_1,k_2}(b_a).$$

Y functions

$$Y_{2j}(\zeta,u_a)=\frac{\hat{W}_{-j,j}\hat{W}_{-j-1,j+1}}{\hat{W}_{-j-1,-j}\hat{W}_{j,j+1}}(\zeta,u_a),$$

$$Y_{2j+1}(e^{-\frac{\pi i}{2}}\zeta,u_a)=\frac{\hat{W}_{-j-1,j}\hat{W}_{-j-2,j+1}}{\hat{W}_{-j-2,-j-1}\hat{W}_{j,j+1}}(\zeta,u_a).$$

Y-system:

$$Y_s(\zeta e^{\frac{\pi i}{2}},u_a)Y_s(\zeta e^{-\frac{\pi i}{2}},u_a)=\Big(1+Y_{s+1}(\zeta,u_a)\Big)\Big(1+Y_{s-1}(\zeta,u_a)\Big),$$

- u_a fixed arbitrarily, single spectral parameter ζ
- massless limit of Y-system in AdS_3 minimal surface [Alday-Maldacena-Sever-Vieira, Hatsuda-Ito-Sakai-Satoh]

TBA equation

asymptotics:

$$\hat{y}_k(q, u_a, \zeta) \sim (-1)^{\frac{k}{2}} c(\zeta) \exp\left(-\frac{i\delta_k}{\zeta} \int_{q^{(k)}}^{q} P(q')dq'\right).$$

asymptotic behavior of the Y-function $\zeta \to \infty$

$$\log Y_{2k+1}(\zeta, u_a) \sim -\frac{1}{\zeta} \oint_{\gamma_{r-2k}} p(q)dq = -\frac{m_{r-2k}}{\zeta},$$

$$\log Y_{2k}(\zeta, u_a) \sim -\frac{i}{\zeta} \oint_{\gamma_{r+1-2k}} p(q)dq = -\frac{m_{r+1-2k},}{\zeta},$$

$$\zeta = e^{-\theta}, \qquad Y_a(\zeta) = e^{-\epsilon_a(\theta)}$$

TBA equations ($\Longleftarrow$ Y-system+asymptotics [Al. Zamolodchikov])

$$\epsilon_a(\theta) = m_a e^{\theta} - \int_{\mathbf{R}} \frac{L_{a-1}(\theta')}{\cosh(\theta - \theta')} \frac{d\theta'}{2\pi} - \int_{\mathbf{R}} \frac{L_{a+1}(\theta')}{\cosh(\theta - \theta')} \frac{d\theta'}{2\pi},$$

TBA in the UV limit

constant solution $\epsilon_a(\theta) \to \epsilon_a^\star$ $(\theta \to -\infty)$

$$Y_a^\star = \frac{\sin\left(\frac{\pi a}{r+3}\right)\sin\left(\frac{\pi(a+2)}{r+3}\right)}{\sin^2\left(\frac{\pi}{r+3}\right)}.$$

effective central charge

$$c_{\mathrm{eff}} = \frac{6}{\pi^2}\sum_{a=1}^{r} m_a \int_{\mathbf{R}} e^{\theta} L_a(\theta) d\theta = \frac{r(r+1)}{r+3}.$$

generalized paramermion $SU(r+1)_2/U(1)^r$ [HISS]

effective central charge and PNP relation

1-cycles with canonical intersection numbers:

$$A_j = \gamma_{2j-1}, \qquad B_j = \sum_{k=j}^{g} (-1)^{k-j} \gamma_{2k},$$

quantum SW periods:

$$\nu_j = \frac{1}{2\pi} \oint_{A_j} P(q) dq, \qquad \nu_{D,j} = i \oint_{B_j} P(q) dq.$$

$$\nu_j = \sum_{n \geq 0} \nu_j^{(n)} \hbar^{2n}, \qquad \nu_{D,j} = \sum_{n \geq 0} \nu_{D,j}^{(n)} \hbar^{2n},$$

the quantum, or NS free energy

$$\frac{\partial F^{\mathrm{NS}}}{\partial \nu_j} = \nu_{D,j}, \qquad j = 1, \cdots, g.$$

$$F^{\mathrm{NS}}(\nu) = \sum_{n \geq 0} F_n^{\mathrm{NS}}(\nu) \hbar^{2n}.$$

$$\mathcal{J}_n = 2(n-1)F_n(\nu) + \sum_{j=1}^{g} \nu_j \frac{\partial F_n^{\mathrm{NS}}}{\partial \nu_j}, \qquad n \geq 0.$$

Matone's relation [Matone, Eguchi-Yang, Sonnenshein-Theisen-Yankielowicz]

$$\mathcal{J}_0 = -2F_0^{\mathrm{NS}} + \sum_{j=1}^{g} \nu_j \frac{\partial F_0}{\partial \nu_j} \propto E$$

quantum Matone's relation

$$\mathcal{J}_1 = \sum_{j=1}^{g} \left(\nu_j^{(0)} \nu_{D,j}^{(1)} - \nu_j^{(1)} \nu_{D,j}^{(0)} \right).$$

$$\mathcal{J}_1 = -\frac{c_{\mathrm{eff}}}{12}.$$

PNP(perturbative- non-perturbative) relation [Codesido-Marino]

$$\mathcal{J} = \alpha E + \beta \hbar^2$$

for genus $g = 1$ curve.

Wall-crossing and TBA

generic potential: turning points are complex.

real mass$\rightarrow$ complex mass

$$m_a = |m_a| e^{i\phi_a}, \qquad a = 1, \cdots, r.$$

$$\widetilde{\epsilon}_a(\theta) = \epsilon_a(\theta - i\phi_a), \qquad \widetilde{L}_a(\theta) = L_a(\theta - i\phi_a).$$

TBA system:

$$\widetilde{\epsilon}_a = |m_a| e^{\theta} - K_{a,a-1} \star \widetilde{L}_{a-1} - K_{a,a+1} \star \widetilde{L}_{a+1}$$

$$K_{r,s} = \frac{1}{2\pi} \frac{1}{\cosh\left(\theta + i(\phi_s - \phi_r)\right)}$$

$$(K \star f)(\theta) = \int_{\mathbf{R}} K(\theta - \theta') f(\theta') d\theta'$$

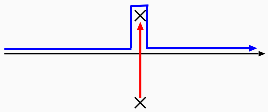
This TBA formula is valid for

$$|\phi_a - \phi_{a\pm1}| < \frac{\pi}{2}$$

When

$$\phi_2 - \phi_1 > \frac{\pi}{2},$$

the kernel picks the pole at $\theta' = \theta - i\phi_2 + i\phi_1 - \frac{\pi i}{2}$.



TBA equations are modified to

$$\widetilde{\epsilon}_1(\theta) = |m_1|e^{\theta} - K_{1,2} \star \widetilde{L}_2 - L_2\left(\theta - i\phi_1 - \frac{i\pi}{2} + i\delta\right),$$

$$\widetilde{\epsilon}_2(\theta) = |m_2|e^{\theta} - K_{2,1} \star \widetilde{L}_1 - L_1\left(\hat{\theta} - i\phi_2 + \frac{i\pi}{2} - i\delta\right).$$

Wall crossing phnenomena of the TBA equations [GMN, AMSV, Toledo unpublished]

3-term TBA

We can transform this 2-term TBA into 3-term TBA equations.

Y-functions $\tilde{\epsilon}_a^{\mathrm{n}}(\theta - i\phi_a) = -\log Y_a^{(\mathrm{n})}(\theta)$ $(a = 1, 2, 12)$

cycles γ_1, γ_2 and $\gamma_{12} = \gamma_1 + \gamma_2$

$$Y_1^{\mathrm{n}}(\theta) = \frac{Y_1(\theta)}{1 + Y_2\left(\theta - \frac{i\pi}{2}\right)}, \qquad Y_2^{\mathrm{n}}(\theta) = \frac{Y_2(\theta)}{1 + Y_1\left(\theta + \frac{i\pi}{2}\right)},$$

$$Y_{12}^{\mathrm{n}}(\theta) = \frac{Y_1(\theta) Y_2\left(\theta - \frac{i\pi}{2}\right)}{1 + Y_1(\theta) + Y_2\left(\theta - \frac{i\pi}{2}\right)}.$$

TBA equations:

$$\widetilde{\epsilon}_1(\theta) = |m_1| e^{\theta} - K_{1,2} \star \widetilde{L}_2 - K_{1,12}^{+} \star \widetilde{L}_{12},$$
$$\widetilde{\epsilon}_2(\theta) = |m_2| e^{\theta} - K_{2,1} \star \widetilde{L}_1 - K_{2,12} \star \widetilde{L}_{12},$$
$$\widetilde{\epsilon}_{12}(\theta) = |m_{12}| e^{\theta} - K_{12,1}^{-} \star \widetilde{L}_1 - K_{12,2} \star \widetilde{L}_2.$$

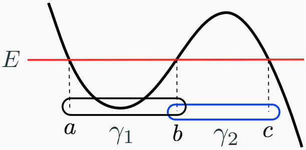
$m_{12} = m_1 - i m_2 = |m_{12}| e^{i\phi_{12}}$ [GMN, HISS]

Example: cubic potential

cubic potential

cubic potential

$$V(x) = \frac{\kappa x^2}{2} - x^3$$



m_1, m_2 are represented by the elliptic integral.

TBA equations

$$\epsilon_1(\theta) = m_1 e^\theta - \int_{\mathbf{R}} \frac{\log\left(1 + e^{-\epsilon_2(\theta')}\right)}{\cosh(\theta - \theta')} \frac{d\theta'}{2\pi},$$

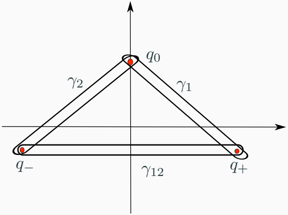
$$\epsilon_2(\theta) = m_2 e^\theta - \int_{\mathbf{R}} \frac{\log\left(1 + e^{-\epsilon_1(\theta')}\right)}{\cosh(\theta - \theta')} \frac{d\theta'}{2\pi}.$$

- $c_{\text{eff}} = \frac{6}{5}$, PNP relation $\mathcal{J} = -\frac{2E}{15} - \frac{\hbar^2}{10}$ [Codesido-Marino]
- quantum periods and TBA system (numerical check)

PT-symmetric Hamiltonian

PT-symmetric Hamiltonian [Bender-Boettcher 1997]

$$\hat{H} = \frac{1}{2}\hat{p}^2 + i\hat{q}^3 - i\lambda\hat{q}.$$



- PT-symmetry: parity $q \to -q+$ complex conjugation
- $\lambda \leq 0$ real, positive discrete spectrum
- $\lambda > 0$ and large, PT-symmetry is broken due to non-perturbative effect

$\boxed{\lambda = 1}$ three complex turning points

$$\Pi_{\gamma_1} = \frac{1}{2}\Pi_{\rm p} - \frac{i}{2}\Pi_{\rm np}, \ \Pi_{\gamma_2} = \Pi_{\gamma_1}^*, \ \Pi_{\gamma_{12}} = \Pi_{\gamma_1} + \Pi_{\gamma_2}$$

3-term TBA

$$\begin{aligned}
\widetilde{\epsilon}_1(\theta) &= |m_1|e^{\theta} - K_{1,2} \star \widetilde{L}_2 - K^{+}_{1,12} \star \widetilde{L}_{12}, \\
\widetilde{\epsilon}_2(\theta) &= |m_2|e^{\theta} - K_{2,1} \star \widetilde{L}_1 - K_{2,12} \star \widetilde{L}_{12}, \\
\widetilde{\epsilon}_{12}(\theta) &= |m_{12}|e^{\theta} - K^{-}_{12,1} \star \widetilde{L}_1 - K_{12,2} \star \widetilde{L}_2.
\end{aligned}$$

with $\phi_1 = -\alpha$, $\phi_2 = \frac{\pi}{2} + \alpha$, $\phi_{12} = 0$.

quantum periods:

$$\frac{1}{\hbar} s(\Pi_{\gamma_1})(\hbar) = -i\widetilde{\epsilon}_1(\theta + i\frac{\pi}{2} - i\alpha), \quad \frac{1}{\hbar} s(\Pi_{\gamma_1})(\hbar) = -i\widetilde{\epsilon}_2(\theta - i\frac{\pi}{2} + i\alpha)$$

exact quantization condition

$$2\cos\left(\frac{1}{2\hbar} s(\Pi_{\rm p})(\hbar)\right) + e^{-\frac{1}{2\hbar} s(\Pi_{\rm np})(\hbar)} = 0.$$

[DDP, Alvarez-Casares,Delabaere-Trinh]

ODE/IM and PT-symmetric Hamiltonian

$\boxed{\lambda = 0}$ $H = \frac{p^2}{2} + iq^3$ $q_0 = i$, $q_+ = e^{-\frac{pii}{6}}$, $q_- = -e^{\frac{\pi}{6}}$, $\alpha = \frac{\pi}{3}$

- Z_3-symmetry: $\widetilde{\epsilon}_a(\theta) = \epsilon(\theta)$
- periodicity $\epsilon(\theta + \frac{5\pi}{3}) = \epsilon(\theta)$

3-term$\Longrightarrow$ single TBA [Dorey-Tateo]

$$\epsilon(\theta) = \frac{\sqrt{6\pi}\Gamma(1/3)}{3\Gamma(11/6)} e^{\theta} + \int_{\mathbf{R}} \Phi(\theta - \theta') L(\theta') d\theta', \quad \Phi(\theta) = \frac{\sqrt{3}}{\pi} \frac{\sinh 2\theta}{\sinh 3\theta}$$

TBA for Yang-Lee edge singularity ($c_{\mathrm{eff}} = \frac{6}{5}$) [Al Zamolodchikov]

Numerical check

PT cubic oscillator with $\hbar = \sqrt{2}$

n	E_n^{num}	E_n^{TBA}	E_n^{p}	E_n^{WKB}
0	$1.156\,267\,071\,988$	$1.156\,267\,071\,988$	$1.134\,513\,239\,424$	$1.094\,269\,500\,533$
1	$4.109\,228\,752\,810$	$4.109\,228\,752\,806$	$4.109\,367\,351\,095$	$4.089\,496\,119\,273$
2	$7.562\,273\,854\,979$	$7.562\,273\,854\,971$	$7.562\,273\,170\,784$	$7.548\,980\,437\,586$

- E_n^{num} : complex dilatation+Rayleigh-Ritz [Yaris-Bendler-Lovett-Bender-Fedders 1978]
- E_n^{TBA} : TBA+exact quantization condition
- E_n^{p}: TBA+Bohr-Sommerfeld quantization condition
- E_n^{WKB} : Bohr-Sommerfeld approximation

Conclusions and outlook

Conclusions and outlook

We have provided an efficient approach to solve the spectral problem for arbitrary polynomial potentials in one-dimensional Quantum Mechanics. The solution takes the form of a TBA system for the (resummed) quantum periods which generalizes the ODE/IM correspondence

- including angular momentum potential $\frac{\ell(\ell+1)}{x^2}$ [DT, Bahzanov-Lukyanov-Zamolodchikov]
- higher order ODEs [Dorey-Dunning-Masoero-Suzuki-Tateo, Sun, Ito-Locke]
- integrable models
  higher integral of motions [BLZ, Bazhanov-Hibberd-Koroshkhin]
  massive ODE/IM [Lukyanov-Zamolodchikov,Dorey et al. , Ito-Locke, Adamopoulou-Dunning, Negro,Ito-Shu]
- Argyres-Douglas theory [Ito-Shu, Grassi-Marino]
  quantum periods and free energy, resurgence structure