Boundary Scattering and Non-Invertible Symmetries in 1+1 Dimensions

Satoshi Yamaguchi (University of Osaka)

Based on S. Shimamori, SY, arXiv:2504.08375

Introduction

Non-invertible symmetry

A class of generalized symmetry "Does not have inverse"

A nice review [Shao, 2308.00747]

Found in many QFTs:

2 dim: [Verlinde], …, [Frohlich, Fuchs, Runkel, Schweigert],… [Bhardwaj, Tachikawa], [Chang, Lin, Shao, Wang, Yin],…

Higher dim:

[Koide, Nagoya, SY], [Kaidi, Ohmori, Zheng], [Choi, Cordova, Hsin, Lam, Shao],…

Any application ?

[Copetti, Cordova, Komatsu 24]

Application for

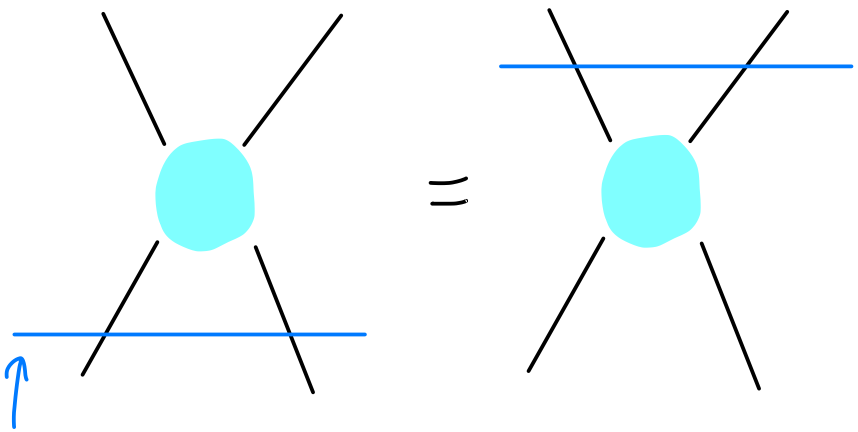
Scattering Amplitude

[Copetti, Cordova, Komatsu 24]

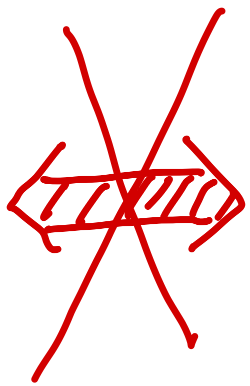
2D integrable system with non-invertible symmetry

Assume unitarity and Yang-Baxter equation

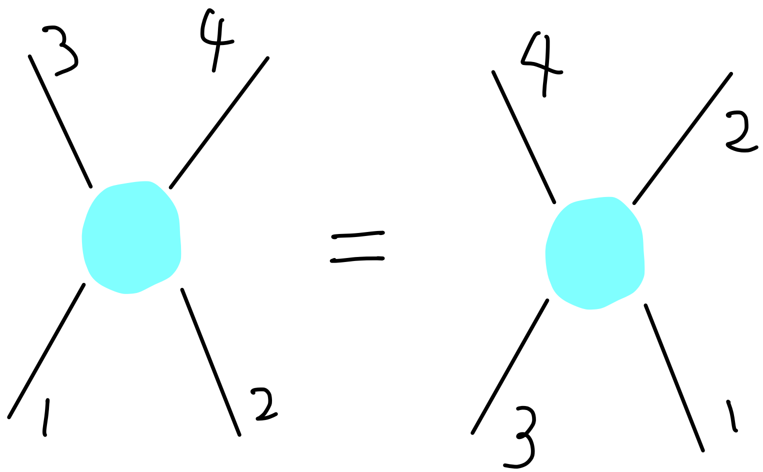
Ward-Takahashi identity for non-invertible symmetry



topological defect



Crossing symmetry



Incompatible!

[Copetti, Cordova, Komatsu 24]

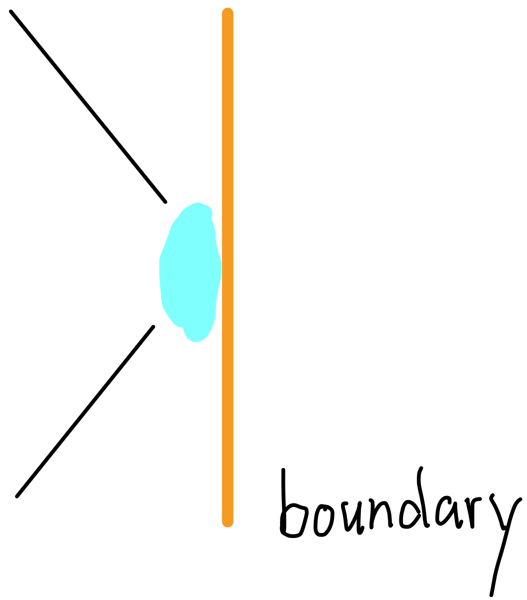
Crossing symmetry must be modified !

Old: $$S^{ab}_{dc}(\theta) = S^{bc}_{ad}(i\pi - \theta)$$

New: $$S^{ab}_{dc}(\theta) = \sqrt{\frac{d_a d_c}{d_b d_d}}\, S^{bc}_{ad}(i\pi - \theta)$$

Detail will be explained later.

[Shimamori, SY] considered boundary scattering with non-invertible symmetry.

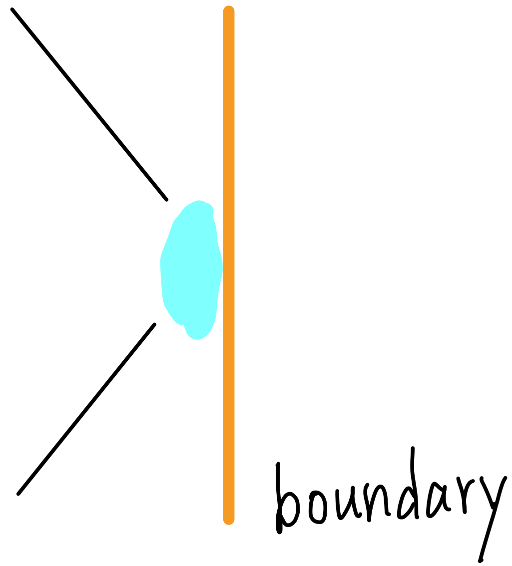


Why boundary?

My long term interest:

- D-branes in string theory (boundary of worldsheet).
- Understanding anomalies.
- …

Result [Shimamori, SY]



Boundary S-matrix

$$\Rightarrow \quad R^{b}_{ca}(\theta)$$

Old:

$$R^{b}_{ca}\left(\frac{i\pi}{2}-\theta\right) = \sum_{d} S^{ba}_{cd}(2\theta)\, R^{d}_{ca}\left(\frac{i\pi}{2}+\theta\right)$$

New:

$$R^{b}_{ca}\left(\frac{i\pi}{2}-\theta\right) = \sum_{d} \sqrt{\frac{d_d}{d_b}}\, S^{ba}_{cd}(2\theta)\, R^{d}_{ca}\left(\frac{i\pi}{2}+\theta\right)$$

Why crossing symmetry is modified?

S-matrix

Reconsider !

LSZ formula

Crossing symmetry

LSZ formula fails to be true in some cases.

Plan

- LSZ formula
- LSZ formula fails?
- SSB of non-invertible symmetry and modified LSZ formula
- Boundary scattering
- Summary and discussion

LSZ formula

Outline

Particle

elementary, composite, soliton,…

State (One particle state)

$|k, i\rangle$

momentum

other label

$\Downarrow$

S-matrix

Fields

elementary, composite, may not be written as a functional of elementary fields,…

Operator

$\phi(x)$

$\Downarrow$

Correlation function

LSZ formula

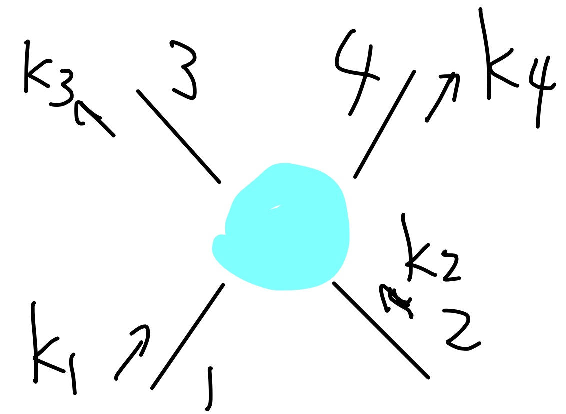
(Non-perturbative !)

Setup

d-dim, scalar particles, 1, 2, 3, 4

elementary, composite, soliton,... m_1, m_2, m_3, m_4

scattering $12 \to 34$



S-matrix element

$$S_{12\to34}(s,t)$$

$$s := -(k_1+k_2)^2$$

$$t := -(k_1-k_3)^2$$

LSZ formula. (See eg. [Peskin, Schroeder Ch7])

Assumption

$\exists\ \phi_i(x)$

$i = 1, 2, 3, 4$

Local operators

elementary, composite, may not be written as a functional of elementary fields,…

s.t.

$$\langle k_i, i | \phi_i(x) | 0 \rangle \neq 0$$

(This condition is satisfied unless you have any reason for vanishing.)

LSZ formula. (See eg. [Peskin, Schroeder Ch7])

$$\phi_i(x) \xrightarrow{\text{Fourier}} \tilde{\phi}_i(k)$$

2pt function

$$\langle \tilde{\phi}_i(k)\, \tilde{\phi}_i(k') \rangle = \Delta_i(k^2)\, \frac{1}{i}\, (2\pi)^d\, \delta^d(k+k')$$

$$\Delta_i(k^2) \underset{\text{pole}}{\sim} \frac{R_i}{k^2+m_i^2} \qquad (R_i > 0)$$

$$S_{12\to 34}(s,t) = \lim_{\substack{\text{on-shell} \\ k_i^2 \to -m_i^2}} \underbrace{\frac{i\sqrt{R_1}}{\Delta_1(k_1^2)}}\ \underbrace{\frac{i\sqrt{R_2}}{\Delta_2(k_2^2)}}\ \underbrace{\frac{i\sqrt{R_3}}{\Delta_3(k_3^2)}}\ \underbrace{\frac{i\sqrt{R_4}}{\Delta_4(k_4^2)}}_{\text{"amputate"}}$$

$$\times \langle \tilde{\phi}_1(-k_1)\, \tilde{\phi}_2(-k_2)\, \tilde{\phi}_3(k_3)\, \tilde{\phi}_4(k_4) \rangle$$

Crossing symmetry

$$S_{12\to 34}(s,t) = \lim_{\substack{\text{on-shell} \\ k_i^2 \to -m_i^2}} \frac{i\sqrt{R_1}}{\Delta_1(k_1^2)} \frac{i\sqrt{R_2}}{\Delta_2(k_2^2)} \frac{i\sqrt{R_3}}{\Delta_3(k_3^2)} \frac{i\sqrt{R_4}}{\Delta_4(k_4^2)} \langle \tilde{\phi}_1(-k_1)\, \tilde{\phi}_2(-k_2)\, \tilde{\phi}_3(k_3)\, \tilde{\phi}_4(k_4) \rangle$$

Invariant under exchange:

$$2 \Leftrightarrow 3 \,, \quad k_2 \Leftrightarrow -k_3 \qquad (\Rightarrow \quad s \Leftrightarrow t\,)$$

$$\Downarrow$$

Crossing symmetry

$$\boxed{S_{12\to 34}(s,t) = S_{13\to 24}(t,s)}$$

(Analytically continued)

LSZ formula fails?

Assumption: $^{\exists}\phi_i(x)$ s.t. $\langle k, i | \phi_i(x) | 0 \rangle \neq 0$

$\Updownarrow$ Not satisfied

$$\boxed{{}^{\forall}\phi_i(x) : \text{local operator}, \quad \langle k, i | \phi_i(x) | 0 \rangle = 0}$$

Does this happen ?

Yes !

Example

1+1 dim, finite global symmetry is sponeteously broken,

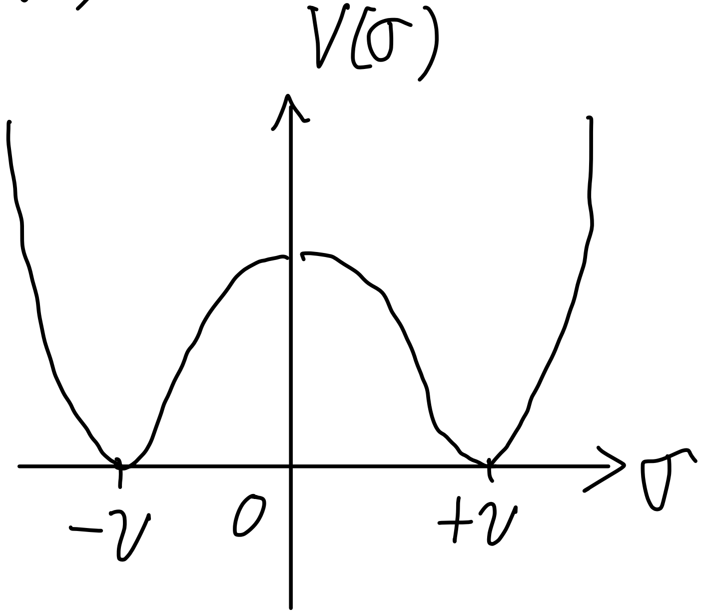
$\Rightarrow$ **Kink solitons** satisfy

$$^{\forall}\phi_i(x) : \text{local operator}, \quad \langle k, i | \phi_i(x) | 0 \rangle = 0$$

Example

(1+1) dim real scalar $\sigma(x)$

$$S = \int d^2x \left[-\frac{1}{2}(\partial\sigma)^2 - V(\sigma) \right]$$



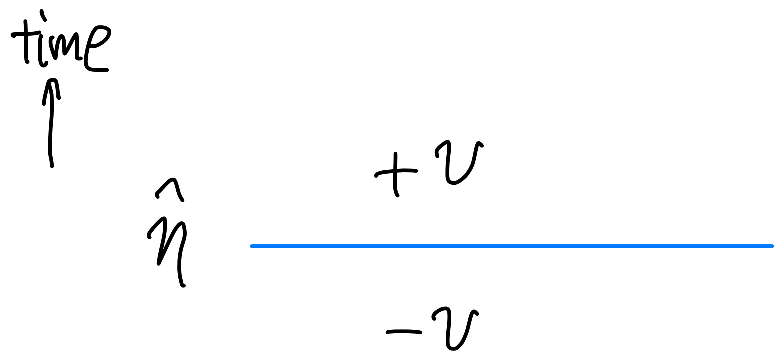
$\mathbb{Z}_2$ Symmetry η

$$\hat{\eta}\,\hat{\sigma}(x)\,\hat{\eta}^{-1} = -\hat{\sigma}(x)$$

spontaneously broken $\Rightarrow$ Two vacua $|+v\rangle, |-v\rangle$

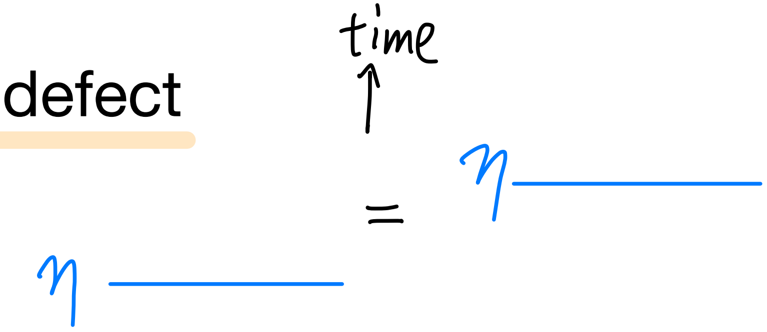
$$\hat{\eta}\,|-v\rangle = |+v\rangle$$

$$\hat{\eta}\,|+v\rangle = |-v\rangle$$

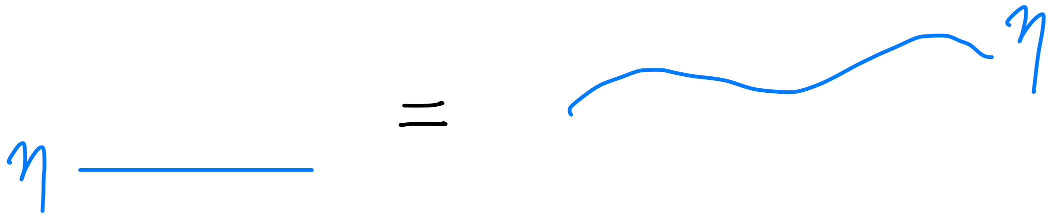


Symmetry $\Rightarrow$ Topological defect

$$\hat{H}\hat{\eta} = \hat{\eta}\hat{H}$$



$\rightsquigarrow$ invariant under continuous deformation



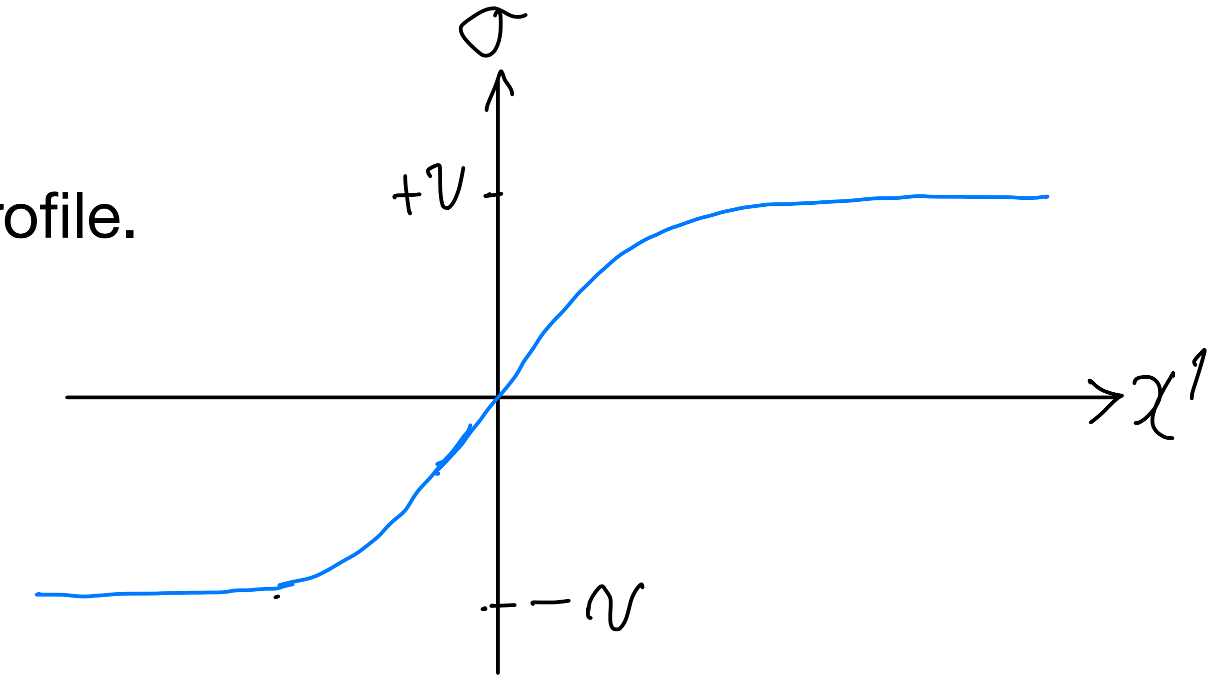
An explanation:
In curved space-time, no canonical time slice, but η is conserved.

$\Rightarrow$ Invariant in any time slice.

Kink soliton

Static, finite energy σ(x) profile.

$$x^1 \to -\infty \ , \ \sigma(x) \to -v$$
$$x^1 \to +\infty \ , \ \sigma(x) \to +v$$



Smallest energy state = stable

Kink (soliton)

- a discrete symmetry is spontaneously broken, kink solitons can exist.
- In 1+1 dim, a kink soliton is a particle.

A kink soliton does not satisfy an assumption of LSZ formula, i.e.

$$|0\rangle := |-v\rangle$$

$\forall\, \phi(x)$ local operator

$$\langle \text{kink} | \phi(x) | 0 \rangle = 0$$

∵ $\phi(x)$ cannot change $|0\rangle = |-v\rangle$ in $x^1 \to \infty$

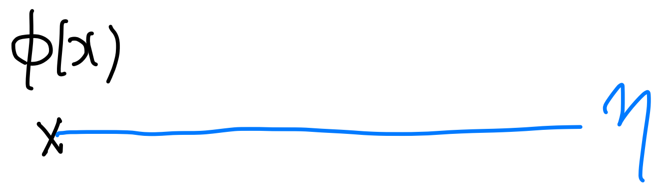
$\langle \text{kink}|$ $\quad -v \quad\quad /\!/\quad\quad +v$

$\times$ $\phi(x)$ $\quad\updownarrow$ overlap vanish

$|0\rangle$ $\quad -v \quad\quad\quad\quad -v \quad\quad \longrightarrow x^1$

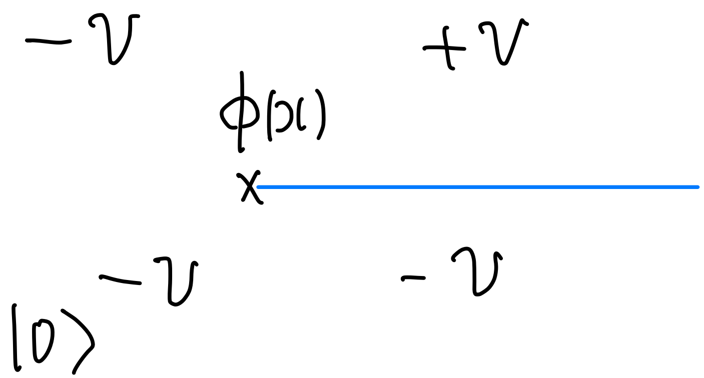
Any alternative?

"Non-genuine local operator"



topological defect of the symmetry

Non-local, but non-local part is topological.



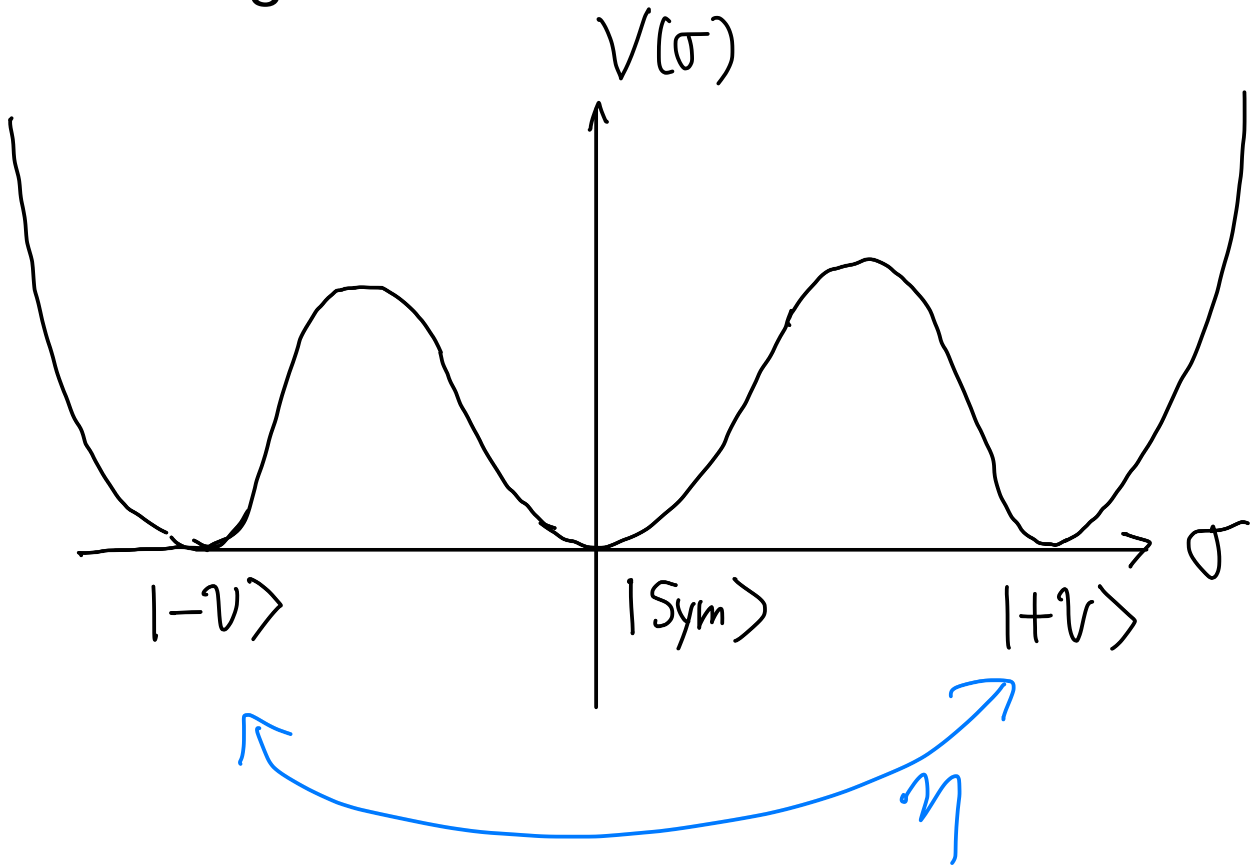
← Includes kink soliton

⟹ reconsider LSZ formula

SSB of non-invertible symmetry and modified LSZ formula

Example: “tricritical Ising CFT, gapped deformation by $\phi_{1,3}$

Skematic figure:



3 degenerate vacua

How about $|\mathrm{Sym}\rangle$?

It cannot explained by a conventional symmetry.

This degeneracy can be explained "non-invertible symmetry."

Operators $\quad 1, \ \hat{\eta}, \ \hat{\mathcal{N}}$

"Ising fusion algebra"

$$\hat{\eta}^2 = 1$$

$$\hat{\mathcal{N}}\hat{\eta} = \hat{\eta}\hat{\mathcal{N}} = \hat{\mathcal{N}}$$

($\hat{\mathcal{N}}$ does not have inverse)

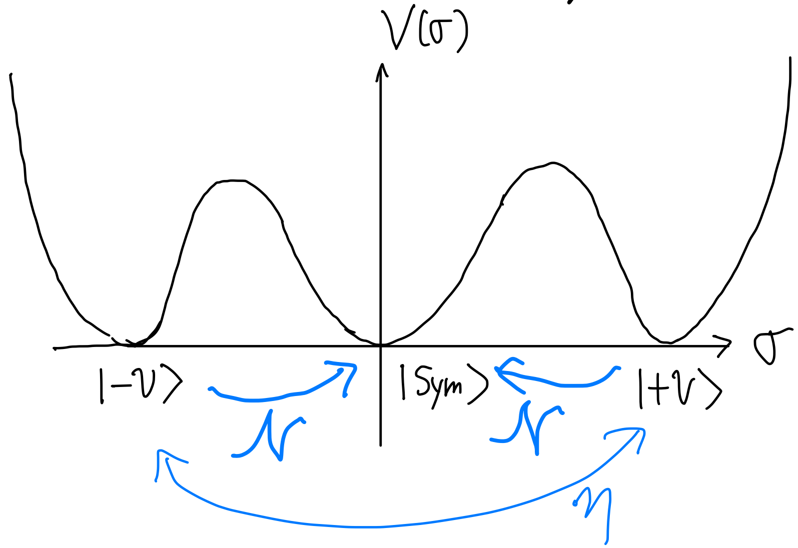
$$\hat{\mathcal{N}}^2 = 1 + \hat{\eta}$$

all commute with $\hat{H}$

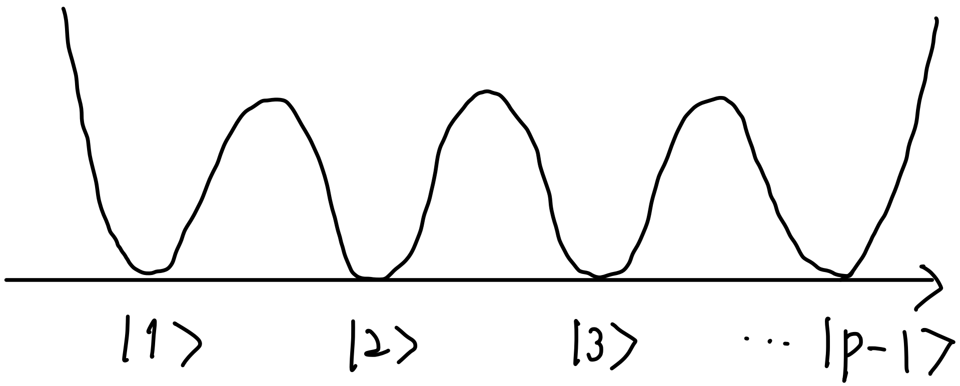
$\Downarrow$

In a gapped phase, at least 3 vacua $\quad |+v\rangle, \ |-v\rangle, \ |\mathrm{Sym}\rangle$

$$\hat{\mathcal{N}}|+v\rangle = \hat{\mathcal{N}}|-v\rangle = |\mathrm{Sym}\rangle$$



Generalization: SSB of $\mathcal{A}_p$ (a fusion category symmetry)



Topological lines

L_a

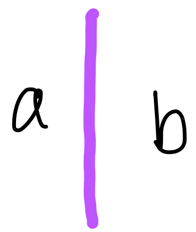
$a = 1, \cdots, p-1$

Reference vacuum

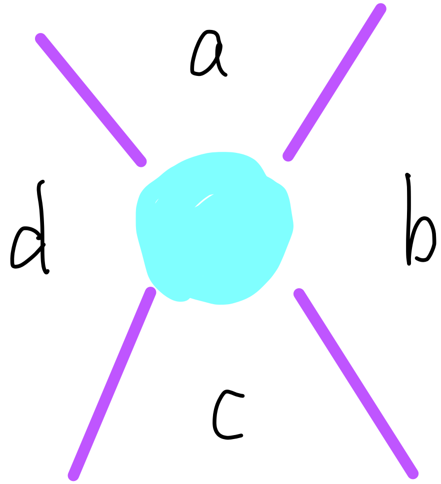
$|0\rangle := |1\rangle \quad \Longrightarrow \quad |a\rangle = L_a |0\rangle$

Kink connecting two vacua $\quad a, b \quad (a = b \pm 1)$

K_{ab} , Same mass due to $\mathcal{A}_p$ symmetry.



Scattering of kinks

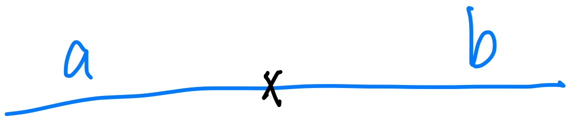


$\Rightarrow S^{ab}_{dc}(\theta)$

Relative rapidity
Lorentz invariant kinematical parameter

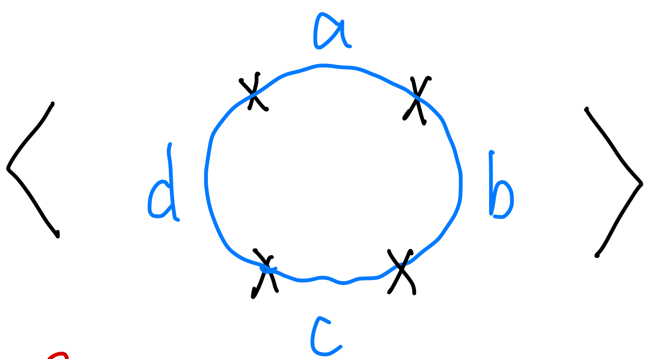
$\phi_{ab}(x)$: Non-genuine local op.

$\parallel$



$$\langle K_{ab} | \phi_{ab}(x) | 0 \rangle \neq 0$$

$\Rightarrow$



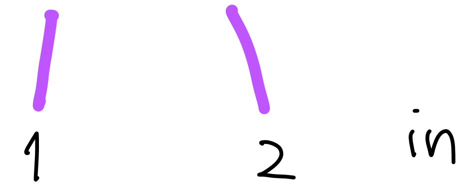
will be related to S^{ab}_{dc} by “LSZ formula”

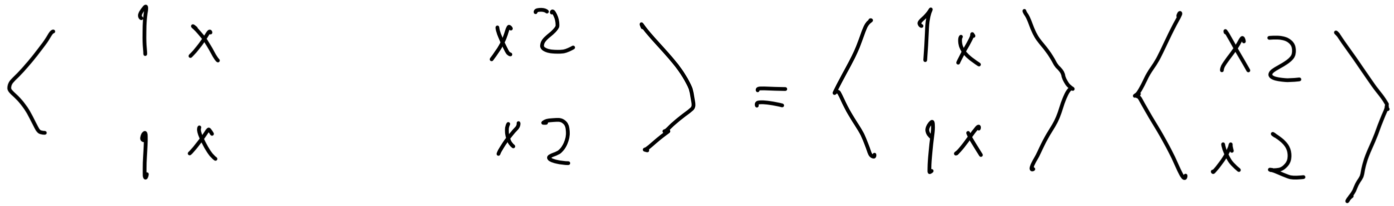
expectation value in $|0\rangle$

How to **amputate** ?

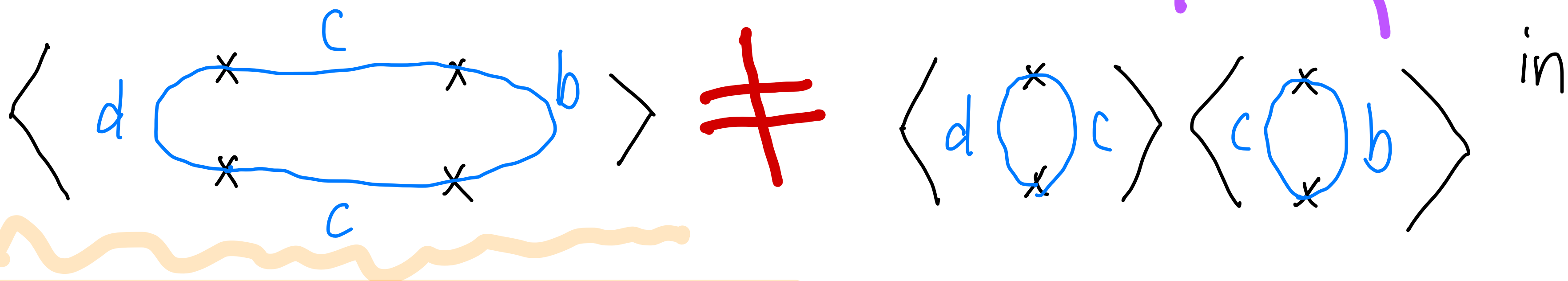
Amputate

Conventional LSZ formula,
genuine local operators





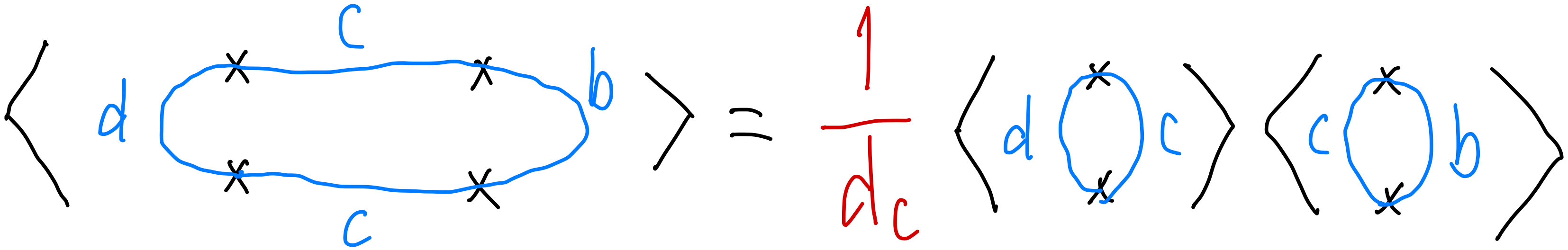
Here, non-genuine local operators



This should be used to amputate!

Amputate

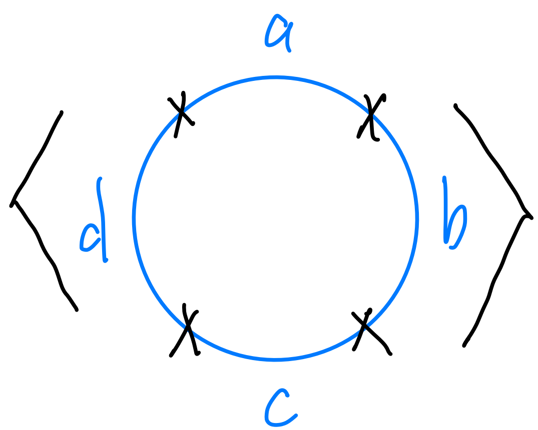
Relation is determined only by the symmetry.



"quantum dimension": determined only by symmetry (independent of dynamics).

Modified LSZ formula

$$S^{a\,b}_{d\,c}(\theta) = \sqrt{d_c\, d_a}$$ (Amputate by 2pt functions)



modify the crossing symmetry

Modified crossing relation

$$S^{ab}_{dc}(\theta) = \sqrt{\frac{d_a d_c}{d_b d_d}}\, S^{bc}_{ad}(i\pi - \theta)$$

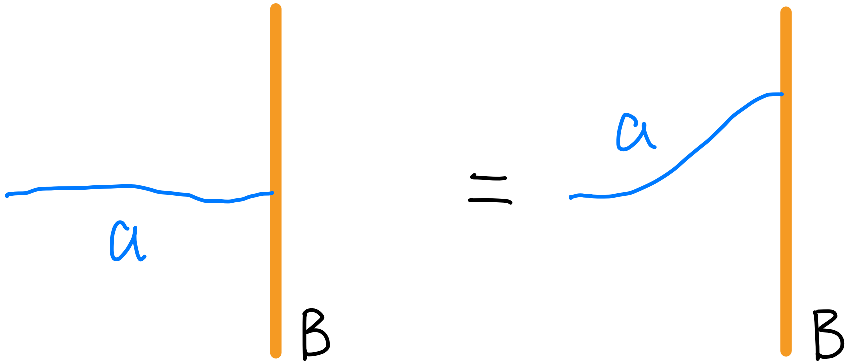
First obtained by [Copetti, Cordova, Komatsu 24]

Boundary scattering

Setup

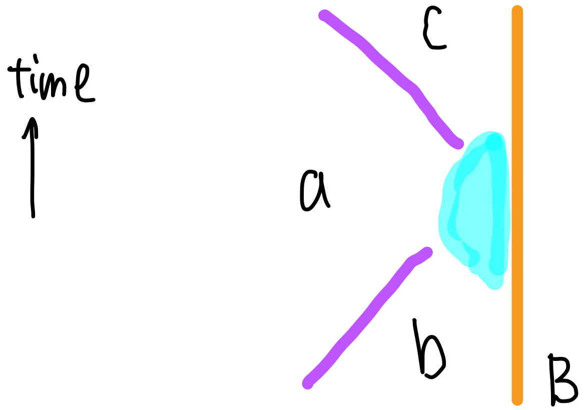
1+1 dim gapped QFT with $\mathcal{A}_p$ symmetry

With weakly symmetric boundary B



Scattering of a kink

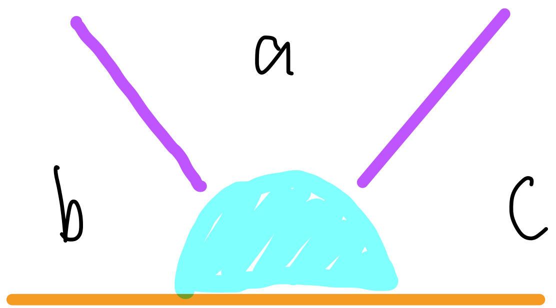
Initiated by [Ghoshal, Zamolodchikov 94]



$$|K_{ab}, \theta, B\rangle_{in} = R^{a}_{bc}(\theta)|K_{ac}, \theta, B\rangle_{out}$$

$$+ \cdots$$

Boundary and particles



$|B\rangle$: boundary state

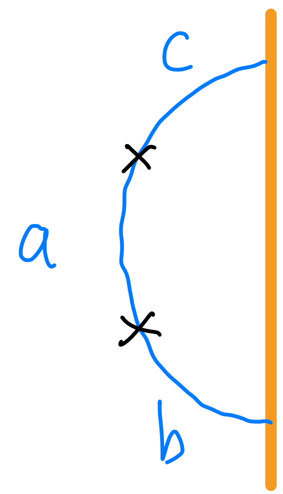
$$|B\rangle = \hat{R}^{a}_{bc}\left(\frac{i\pi}{2}-\theta\right)|K_{ba}; -\theta, K_{ac}, \theta\rangle_{out} + \cdots$$

$\hat{R}^{a}_{bc}$ vs R^{a}_{bc}

Exchange time direction

Precise relation?

both $\hat{R}^{a}_{bc}$ R^{a}_{bc} $\sim$ correlation function



[Ghoshal, Zamolodchikov 94]

(When the symmetry is invertible)

$$\hat{R}^{a}_{bc} = C_B R^{a}_{bc}$$

independent of symmetry

[Shimamori, SY]

$$\hat{R}^{a}_{bc} = \sqrt{d_a F^B_b F^B_c}\, R^{a}_{bc}$$

$\Leftarrow$ Careful consideration of the amputation

only by symmetry

may depends on the details of the theory

[Ghoshal, Zamolodchikov 94]

Unitarity

Bulk integrability

$$\Rightarrow \quad \hat{R}^{b}_{ca}\left(\frac{i\pi}{2}-\theta\right)=\sum_{d} S^{ba}_{cd}(2\theta)\,\hat{R}^{d}_{ca}\left(\frac{i\pi}{2}+\theta\right)$$

[GZ] $\Downarrow$

$$\left(R^{b}_{ca}\left(\frac{i\pi}{2}-\theta\right)=\sum_{d} S^{ba}_{cd}(2\theta)\,R^{d}_{ca}\left(\frac{i\pi}{2}+\theta\right) \right)$$

"Boundary crossing relation"

[Shimamori, SY]

$$\boxed{R^{b}_{ca}\left(\frac{i\pi}{2}-\theta\right)=\sum_{d} \sqrt{\frac{d_d}{d_b}}\; S^{ba}_{cd}(2\theta)\,R^{d}_{ca}\left(\frac{i\pi}{2}+\theta\right)}$$

Comments:

- WT identity, bulk integrability, modified boundary crossing relation lead to a non-trivial consistency condition for S-matrix. [Copetti, Cordova, Komatsu]'s modified solution satisfies this condition.

- Combined with boundary integrability, the solution for $R^{a}_{bc}(\theta)$ is obtained.

Summary and discussion

Conclusion:

Non-invertible symmetry is important also in scattering problems

- SSB of non-invertible symmetry. Scattering of kinks.

  Careful treatment is needed in the derivation of the LSZ formula and its boundary version.

  $\Rightarrow$ Non-trivial modification of known formula

Future prospects:

- Bootstrap approach using modified (boundary) crossing relation. (Eg. [Copetti, Cordova, Komatsu 2408])

- Higher dimensions.

  Failure of the "assumption" is rather common.

  Eg. QED $\langle \text{electron} | \psi(x) | 0 \rangle = 0$

  $\forall \psi(x)$, local, gauge invariant

  $\sim$ IR problems ?

- Callan-Rubakov problem of monopole-electron scattering

  $\Rightarrow$ S wave $\rightarrow$ 2 dim scattering particle with a boundary

  [van Beest, Boyle Smith, Delmastro, Komargodski 23]