Black hole states at finite N

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This talk is based on

- J. Choi, **<u>SC</u>**, S. Kim, J. Lee and S. Lee
  "Finite N black hole cohomologies"
  JHEP **12**, 029 (2024) [arXiv: 2312.16443 [hep-th]]

- **<u>SC</u>**, S. Kim, E. Lee, S. Lee and J. Park
  "Towards quantum black hole microstates"
  JHEP **11**, 175 (2023) [arXiv: 2304.10155 [hep-th]]

- **<u>SC</u>**, D. Jain, S. Kim, V. Krishna, G. Kwon, E. Lee, S. Minwalla and C. Patel
  "Supersymmetric Grey Galaxies, Dual Dressed Black Holes and the Superconformal Index"
  SciPost Phys. **19**, no.3, 072 (2025) [arXiv:2501.17217 [hep-th]]

- **<u>SC</u>**, D. Jain, S. Kim, V. Krishna, E. Lee, S. Minwalla and C. Patel
  "Dual Dressed Black Holes as the end point of the Charged Superradiant instability in N=4 Yang Mills"
  SciPost Phys. **18**, no.4, 137 (2025) [arXiv:2409.18178 [hep-th]]
- **<u>SC</u>**, S. Kim, E. Lee and J. Park "The shape of non-graviton operator for $SU(2)$"
  JHEP **09**, 029 (2024) [arXiv: 2209.12696 [hep-th]]

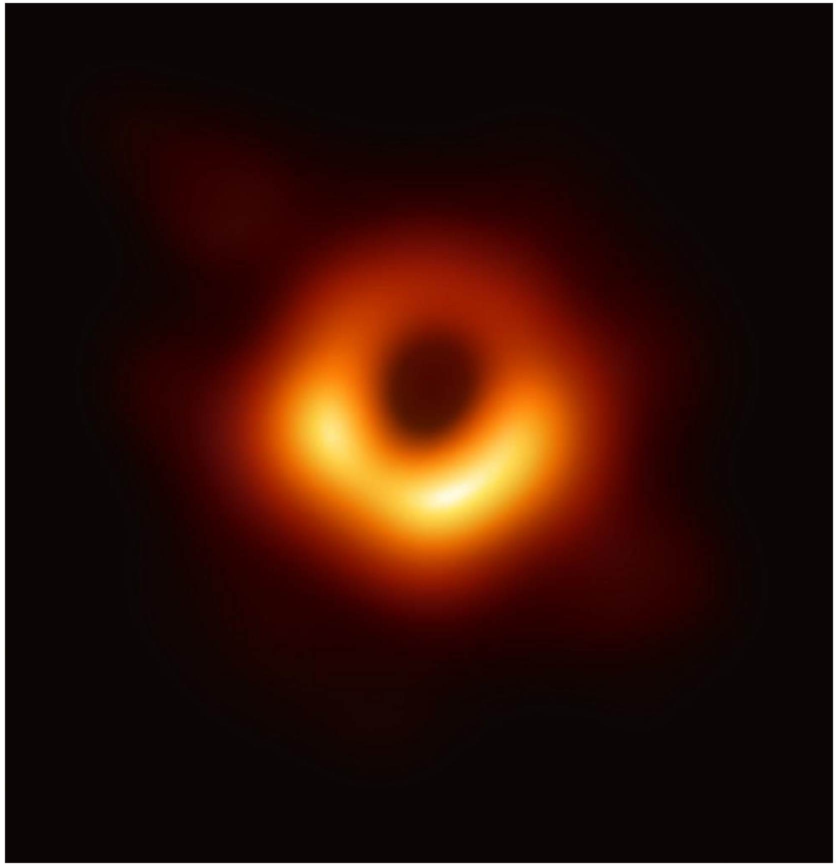
Black holes

- Black holes: fundamental objects in gravity

  - described by a **smooth geometry** of general relativity ~ event horizon

  - encapsulate high energy states in **discrete spectrum** of quantum gravity

  → A theoretical probe to understand important and mysterious properties of quantum gravity



[EHT 19]

Black hole mechanics

- Physical properties of black holes that are satisfied universally

[Bardeen, Carter, Hawking 73]

  - The 1st law: $dE = \frac{\kappa}{8\pi} dA + \Omega\, dJ + \Phi\, dQ$ (for perturbing stationary black holes)

  - The 2nd law: $\frac{dA}{dt} \geq 0$ (area theorem) [Hawking 71]

→ Analogous to **laws of thermodynamics**

→ Can we interpret $T \propto \kappa$ and $S \propto A$ as the thermodynamic **temperature and entropy of black holes**? [Bekenstein 72]

Black hole thermodynamics

- Hawking radiation [Hawking 74]

  - black holes emit thermal radiation at a temperature $T_H = \frac{\hbar}{ck_B}\frac{\kappa}{2\pi}$

- Black holes are thermodynamic objects with **macroscopic entropy.**

$$S_{BH} = \frac{c^3 k_B}{G\hbar}\frac{A}{4}$$

  - ($S_{BH} \fallingdotseq 10^{77} \frac{M^2}{M_{Sun}^2} k_B$ for Schwarzschild black holes)

- **Statistical origin** of the entropy

  - $S_{BH} = k_B \log \Omega$: **number of quantum states** inside the event horizon ?

AdS/CFT correspondence

- **Quantum gravity** in a covariant box obeying the equivalence principle: anti-de Sitter (**AdS**) spacetime ~ vacuum with negative cosmological constant

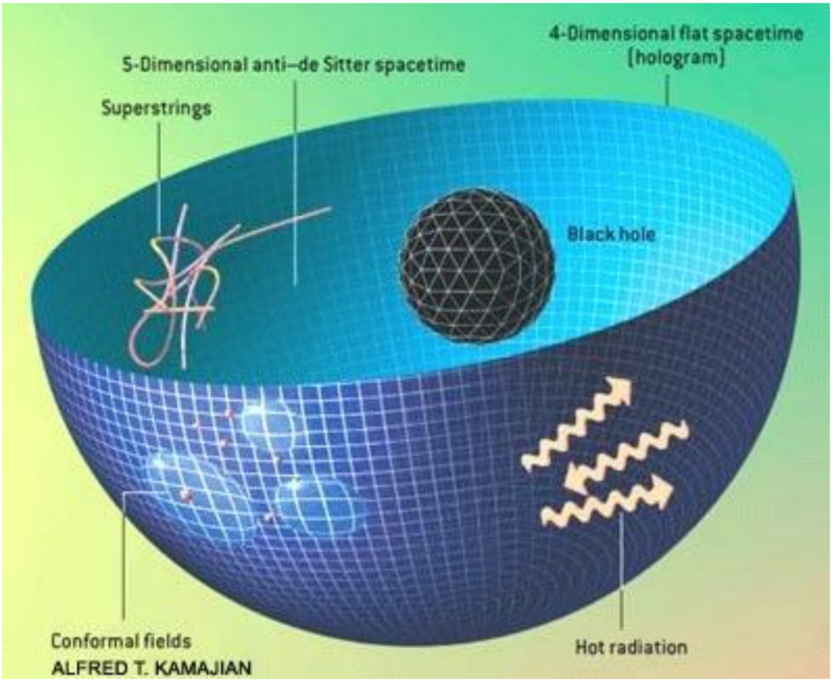
  → Holographically described by
  conformal field theory (CFT) on the **boundary**

Ex) Quantum gravity on asymptotically $\mathrm{AdS}_{4+1} \times S^5$ =

(3+1)-dim. $\mathcal{N}$ =4 supersymmetric $SU(N)$ Yang-Mills theory

[Maldacena 97]

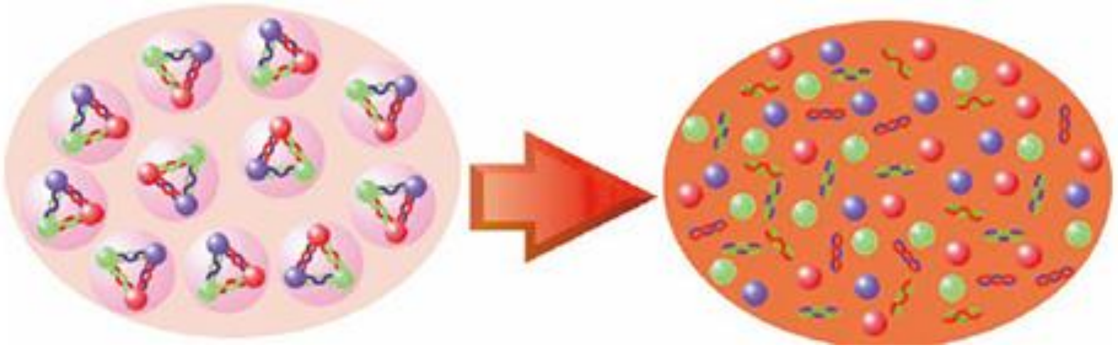
- Semiclassical gravity is emergent when CFT dual
  - has **large degrees of freedom** $N \gg 1$
  - is **strongly coupled** $Ng_{YM}^2 \gg 1$



Black holes ~ deconfined quark-gluon plasma

- **Hawking-Page phase transition** in AdS gravity (ex. AdS_{4+1}) [Hawking, Page 83]
  - low temperature phase: thermal gas of **gravitons**
  - high temperature phase: Schwarzschild **black holes**

- **Confinement-deconfinement transition** in dual gauge theory (on $S^3 \times \mathbb{R}_{time}$) [Witten 98]
  - low temperature phase: **confined** hadrons
  - high temperature phase: **deconfined** quark-gluon plasma



Recent progress on black holes in AdS/CFT

- Recently, there have been lots of progress in understanding various quantum properties of AdS black holes holographically from dual conformal field theories utilizing supersymmetries.

  - accounted for the **Bekenstein-Hawking entropy** of supersymmetric black holes in AdS from the **supersymmetric partition function (~ index)** of CFT duals in various spacetime dimensions
[Cabo-Bizet, Cassani, Martelli, Murthy 18; SC, J.Kim, S.Kim, Nahmgoong 18; Benini, Milan 18; …] [SC, S.Kim 19] [SC, C.Hwang, S.Kim 19; Nian, Pando-Zayas 19; …] [Nahmgoong 19]

  - in AdS_5/CFT_4, **black hole ~ deconfined quark-gluon plasma**: N^2 (~matrix) d.o.f

- A non-trivial **thermodynamic ensemble** exists even at zero temperature.
  - grand-canonical partition function: $Z = Tr[e^{-\Delta\cdot Q-\omega\cdot J}]$ $(\sum\Delta - \sum\omega = 2\pi i)$
    (Q: R-charges, J: angular momenta)
  - exhibits various phase transitions such as **Hawking-Page-like transition** [SC, J.Kim, S.Kim, Nahmgoong 18]

- Certain black holes exhibit **thermodynamic instabilities** reminiscent of dynamical instabilities at finite temperature.
  - thermodynamic instability of over-spinning, over-charged black holes (~**super-radiant instability**)
  - imply existence of **hairy black holes** → **Grey Galaxies, Dual Dressed Black Holes**
[S.Kim, Kundu, E.Lee, J.Lee, Minwalla 23] [Bajaj, Kumar, Minwalla, Mukherjee, Rahaman 24] [SC, Jain, S.Kim, Krishna, E.Lee, Minwalla, Patel 24] [SC, Jain, S.Kim, Krishna, G.Kwon, E.Lee, Minwalla 25]

Black hole microstates

- Graviton (~perturbative) states are well-understood.

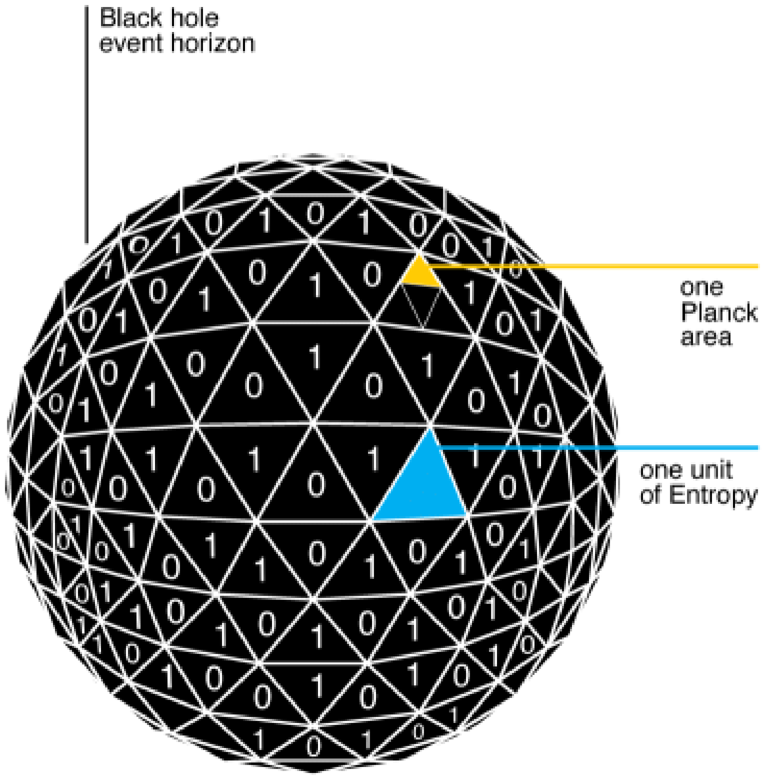
- What's so special about the **black hole microstates** compared to them?

  ~ fundamental degrees of freedom of **quantum gravity**

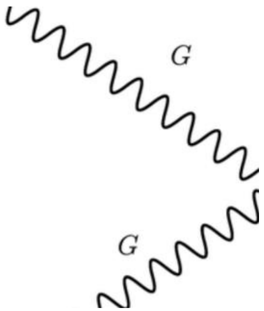
  - **characteristic** properties of black hole states?

- How does **emergent geometric** description (~ event horizon) arise from such black hole microstates?

  - construct **new black hole** solutions?



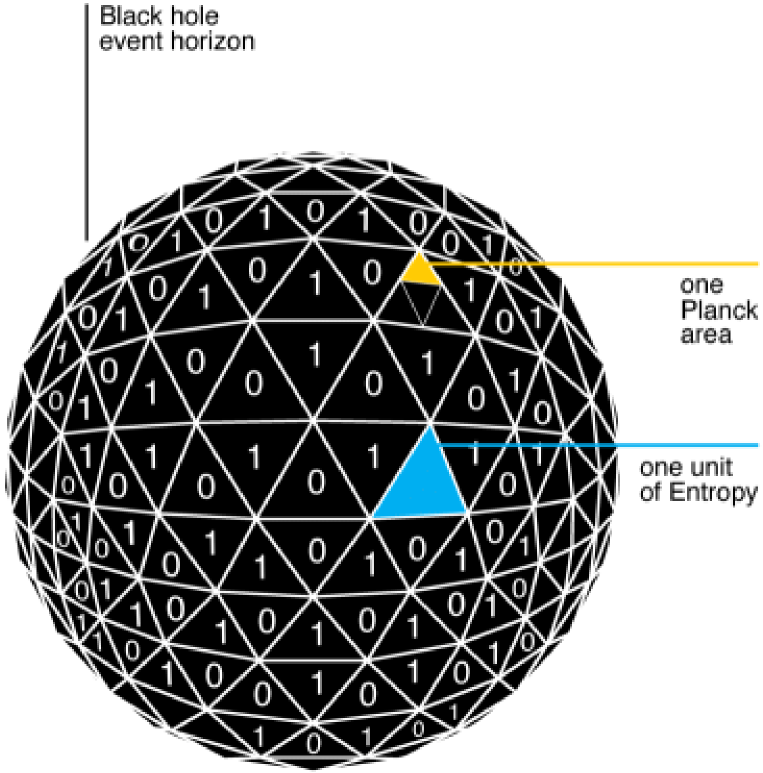
Black hole microstates

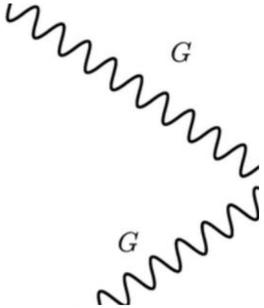


$$\mathrm{tr}[\bar{\phi}^{(m}\bar{\phi}^{n)}]$$

$$v^m{}_n \equiv (\phi^m \cdot \psi_n) - \frac{1}{3}\delta^m_n(\phi^p \cdot \psi_p)$$

Black hole microstates





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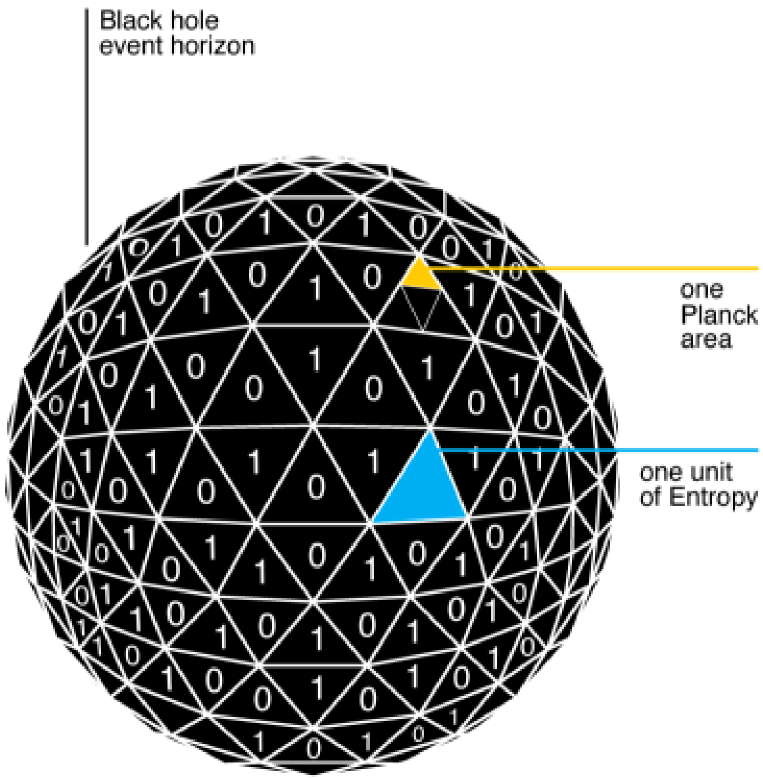
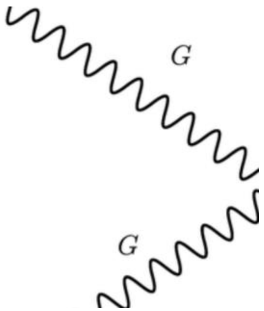
$$O_0 \equiv \epsilon^{p_1 p_2 p_3} v^{m}{}_{p_1} v^{n}{}_{p_2} (\psi_m \cdot \psi_n \times \psi_{p_3})$$

Black hole microstates





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$$O_0 \partial_{+\dot{\alpha}} \partial_{+\dot{\beta}} (\bar{\phi}^m \cdot \bar{\phi}^n)$$

$$O_0 \partial_{+\dot{\alpha}} \partial_{+\dot{\beta}} \partial_{+\dot{\gamma}} (\bar{\phi}^m \cdot \bar{\phi}^n)$$

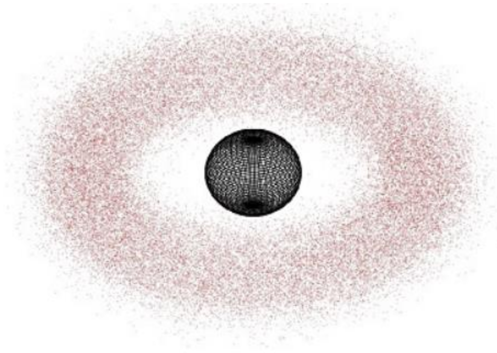
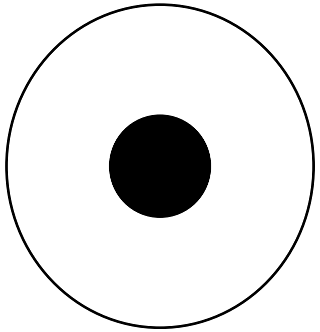


Photo courtesy of Pedro V Cunha.



Black hole states at finite N

- **Strongly coupled** ($Ng_{YM}^2 \gg 1$) gauge theory

  - focus on the **BPS sector**

  → certain operators are **protected**, i.e. do not receive anomalous dimensions at strong coupling (~ gravity).

- **Finite N** gauge theory

  - Newton constant $G \sim {}^{1}/_{N^2}$ in the gravity dual is finite.

  - Even in such a theory, we can distinguish between graviton states and BH states.

  - Finite N BH states can be thought of as quantum black holes in AdS.

  - They exhibit certain properties of **semiclassical black holes**!

BPS operator

- We will study type **IIB string theory on AdS$_5 \times S^5$** holographically from 4D **$\mathcal{N}$=4 $SU(N)$ SYM**.

(gauge-invariant local operators on $\mathbb{R}^4$ ⇔ states on $S^3 \times \mathbb{R}$ ⇔ states on AdS$_5$)

- $\frac{1}{16}$ - BPS operators with $Q \equiv Q^4_-$ and $S \equiv S_4^- = Q^\dagger$: $[Q, O\} = 0$, $[Q^\dagger, O\} = 0$ ⇒ $E = \sum_I R_I + \sum_i J_i$

$$Q^2 = 0 \ , \ (Q^\dagger)^2 = 0 \ , \ \{Q, Q^\dagger\} \sim H - \sum_{I=1}^{3} R_I - \sum_{i=1}^{2} J_i$$

H: scaling dimension
R_I: $SO(6)$ R-charges
J_i: $SO(4)$ angular momenta

- **Free BPS letters**: $\bar{\phi}^m \equiv \Phi^{4m}$, $\psi_m \equiv \Psi_{m+}$, $\bar{\lambda}_{\dot{\alpha}} \equiv \overline{\Psi}^4_{\dot{\alpha}}$, f_{++}, $\partial_{+\dot{\alpha}}$ subject to $\partial_{+\dot{\alpha}}\bar{\lambda}^{\dot{\alpha}} = 0$

$\alpha = \pm, \dot{\alpha} = \dot{\pm}$: $SU(2)_l \times SU(2)_r \cong SO(4)$ rotation symmetry

$m = 1,2,3$: $SU(3)_R \subset SU(3)_R \times U(1) \subset SU(4)_R$ R-symmetry

→ general gauge-invariant local BPS operators in the free theory

BPS cohomology

- $\{Q, Q^\dagger\}$ receives **quantum correction at 1-loop order $O(g_{YM}^2)$**. [Beisert 03, 04]

 - Lots of BPS operators in the free theory acquire non-zero **anomalous dimension at 1-loop level**. [Janik, Trzetrzelewki 07; Chang, Yin 13]

 - Conjecture: no further correction (higher-loop, non-perturbative) [Grant, Grassi, S.Kim, Minwalla 08] [Chang, Yin 13] [Chang, Lin 22]

 - Known examples: SYK models [Chang, Chen, Sia, Yang 24], the D1-D5 system [Chang, Lin, Zhang 25]

 - An explicit **counter-example**: $\frac{1}{16}$-BPS sector of 4d $\mathcal{N}$=4 $SO(7)$ SYM [Gadde, E.Lee, Raj, Tomar 25][Chang, Lin 25]

← a pair of 1-loop BPS operators are lifted at **2-loop** order whose correction was inferred from the generalized **Konishi anomaly** [J.Choi, E.Lee 25]

 - **Complete 1-loop corrections** to Q from the generalized Konishi anomalies [Budzik, Kulp 25]

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- Nevertheless, we have seen that the index correctly captures the **entropy of BPS black holes**.

  - There should be a macroscopic number of **protected operators**, which do not receive anomalous dimensions at strong coupling, responsible for the entropy of BPS black holes.

  - These operators will be holographically dual to the **microstates of BPS black holes**.

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 → We shall construct candidate BPS operators dual to black hole states at **1-loop** order.

- $\frac{1}{16}$ - BPS operators are harmonic forms of $\{Q, Q^\dagger\}$ ⇔ **Q-cohomology** classes $[Q, O\} = 0, \; O \sim O + [Q, \Lambda\}$

 - **Classical Q-action** on free BPS letters: $g_{\rm YM}[\bar\phi^m, \psi_{m+}] \sim D_{+\dot\alpha}\bar\lambda^{\dot\alpha}$

$$[Q, \bar\phi^m] = 0 \; , \; \{Q, \bar\lambda_{\dot\alpha}\} = 0 \; , \; \{Q, \psi_{m+}\} = -ig_{\rm YM}\epsilon_{mnp}[\bar\phi^n, \bar\phi^p]$$

$$[Q, f_{++}] = -ig_{\rm YM}[\psi_{m+}, \bar\phi^m] \; , \; [Q, D_{+\dot\alpha}](\cdots) = -ig_{\rm YM}[\bar\lambda_{\dot\alpha}, (\cdots)\}$$

[Biswas, Gaiotto, Lahiri, Minwalla 07]
[Grant, Grassi, S.Kim, Minwalla 08]

- Our goal is to construct representatives of new cohomology classes which are not of graviton type.

Gravitons vs. black holes

- In the gravity side, the BPS **graviton** states (~all perturbative particle states) can be defined at $\boldsymbol{G = 0}$.

  - One can find local BPS operators in SYM dual to such states at $\boldsymbol{N = \infty}$. (~finite energy states)

  - They persist to exist at arbitrary **finite $\boldsymbol{N}$**.

  → Graviton cohomologies

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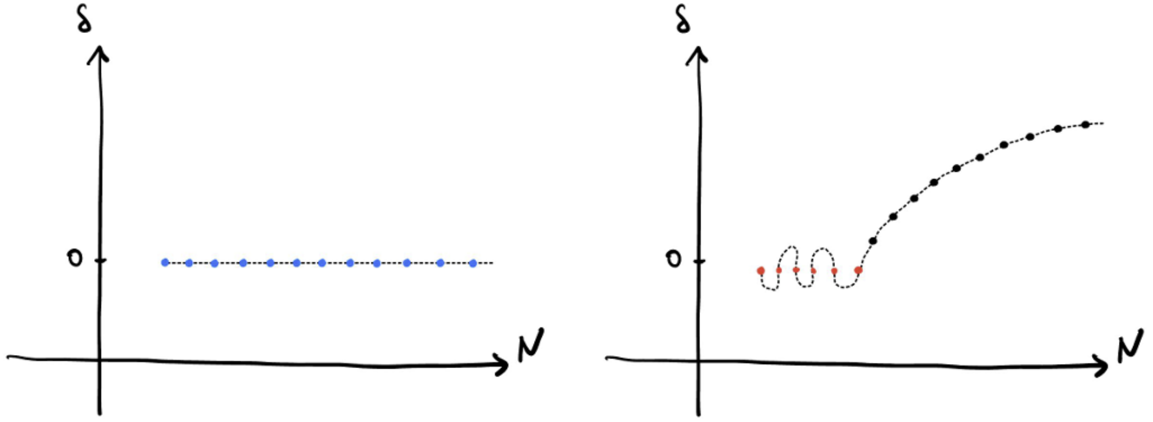
  - They persist to exist at arbitrary **finite $\boldsymbol{N}$**.
  → Graviton cohomologies ~ monotone

- In the gravity side, the BPS **black holes** carry $O(e^{1/G})$ number of states.
  - Correspondingly, there should be $\boldsymbol{O(e^{N^2})}$ number of dual BPS operators in SYM. (~$O(N^2)$ energy states)
  - There should be extra BPS operators at finite N, which are only BPS at **certain $\boldsymbol{N}$** → **trace relation!**
  → Black hole cohomologies ~ fortuitous



[Chang, Lin 24]

Graviton cohomologies

- **Single-trace cohomologies** (Q does not change the trace number):

 - superconformal primaries $|n\rangle = \mathrm{tr}[\bar\phi^{(m_1}\cdots\bar\phi^{m_n)}]$

 - superconformal descendants by Q^m_+, $\bar Q_{m\dot\alpha}$, fermionic generators of $PSU(1,2|3) \subset PSU(2,2|4)$, which commute with our $Q, Q^\dagger$ ($\{Q^m_+, \bar Q_{m\dot\alpha}\} \sim P_{+\dot\alpha}$)

[Kinney, Maldacena, Minwalla, Raju 05]

 - These cohomologies can be arranged into the short supermultiplets of $PSU(1,2|3)$ called $S_{n(\geq 2)}$.

 → absolutely protected in $g_{YM} \to 0 \sim \infty$

 ex) Conformal primaries in S_2:
$$\mathrm{tr}(\bar\phi^{(m}\bar\phi^{n)}) \ , \ \mathrm{tr}(\bar\phi^m\bar\lambda_{\dot\alpha}) \ , \ \mathrm{tr}(\bar\lambda_{\dot+}\bar\lambda_{\dot-}),$$
$$\mathrm{tr}(\bar\phi^m\psi_{n+}) - \tfrac{1}{3}\delta^m_n\mathrm{tr}(\bar\phi^l\psi_{l+}) \ , \ \mathrm{tr}(\bar\lambda_{\dot\alpha}\psi_{m+} - \epsilon_{mnp}\bar\phi^n D_{+\dot\alpha}\bar\phi^p),$$
$$\mathrm{tr}(\bar\phi^m f_{++} - \tfrac{1}{4}\epsilon^{mnp}\psi_{n+}\psi_{p+}) \ , \ \mathrm{tr}(\bar\lambda_{\dot\alpha} f_{++} - \tfrac{2}{3}\psi_{m+}D_{+\dot\alpha}\bar\phi^m + \tfrac{1}{3}\bar\phi^m D_{+\dot\alpha}\psi_{m+})$$

- Multiplying the above cohomologies yields independent **multi-trace cohomologies** at large N.

 (mixing of single and multi-trace operators by the dilatation operator H is suppressed by $\frac{1}{N}$.)

- These correspond to the single and **multi-particle graviton** states in AdS_5. [Chang, Yin 13]

Finite N graviton cohomologies

- The former construction gives cohomologies at arbitrary N.

 - However, they are not all linearly independent.

 - The number of independent cohomologies reduces due to the **trace relations** when the number of fields becomes larger than N.

ex) $SU(2)$:
$$\mathrm{tr}(\bar{\phi}^{(m}\bar{\phi}^{n)}) \sim \mathrm{tr}(X^2)\ ,\ \mathrm{tr}(Y^2)\ ,\ \mathrm{tr}(Z^2)\ ,\ \mathrm{tr}(XY)\ ,\ \mathrm{tr}(YZ)\ ,\ \mathrm{tr}(ZX)$$
$$\mathrm{tr}(X^2)\mathrm{tr}(Y^2) - [\mathrm{tr}(XY)]^2 \sim \mathrm{tr}([X,Y][X,Y]) \sim Q\mathrm{tr}(\psi_3[X,Y])$$

 - **Stringy exclusion principle**: In AdS_5 gravity, gravitons are polarized into **D3-brane giant gravitons** reducing allowed number of states.

 → These BPS states are well understood.

[McGreevy, Susskind, Toumbas 00; Grisaru Myers, Tafjord 00; Hashimoto, Hirano, Itzhaki 00; …]

- **Deconfining phases** of $SU(N)$ gauge theories in the Cardy limit $\mu \to 0$ of $Z(\mu) = \mathrm{Tr}\,(-1)^F e^{-6\mu(R+J)}$:

 - $\frac{R}{N^2}, \frac{J}{N^2} \gg 1 \ \to\ \frac{S}{N^2} \gg 1$ at arbitrary N.

 - Straightforward generalization of the **large black hole physics** to finite N [SC, J.Kim, S.Kim, Nahmgoong 18]

 → Novel finite N cohomologies which are not of graviton type ~ '**black hole cohomologies**'

Black hole index

- Superconformal index ~ Witten index type thermal partition function on $S^3 \times \mathbb{R}$:

$$Z(\Delta_I, \omega_i) = \mathrm{Tr}\left[(-1)^F e^{-\sum_{I=1}^3 \Delta_I R_I} e^{-\sum_{i=1}^2 \omega_i J_i}\right] \quad \sum_I \Delta_I - \sum_i \omega_i = 0$$

  - count the **BPS cohomologies** with $(-1)^F$

[Romelsberger 05]
[Kinney, Maldacena, Minwalla, Raju 05]

  - protected under any **continuous deformation**: can be computed exactly at weak coupling to study strong coupling dynamics (~semiclassical gravity)

  - unitary matrix integral for $SU(N)$ theory: $f(\Delta_I, \omega_i) \equiv 1 - \frac{(1-e^{-\Delta_1})(1-e^{-\Delta_2})(1-e^{-\Delta_3})}{(1-e^{-\omega_1})(1-e^{-\omega_2})}$

$$Z(\Delta_I, \omega_i) = \frac{1}{N!}\int_0^{2\pi} \prod_{a=1}^N \frac{d\alpha_a}{2\pi} \exp\left[\sum_{a\neq b}\sum_{n=1}^\infty \frac{f(n\Delta_I, n\omega_i) - 1}{n} e^{in(\alpha_a - \alpha_b)}\right] \exp\left[(N-1)\sum_{n=1}^\infty \frac{f(n\Delta_I, n\omega_i)}{n}\right]$$

- Count the number of independent graviton cohomologies at finite N.

  → Compute the finite N graviton index.

- Subtract the graviton index from the full index to get the **number of black hole cohomologies** !

BMN cohomologies

- Truncation of our classical cohomology problem into the sector with **no derivative** ~ **BMN matrix model**:

$$\bar{\phi}^m \,, \quad \psi_{m+} \,, \quad f_{++}$$

$$[Q, \bar{\phi}^m] = 0,\ \{Q, \psi_{m+}\} = -i\epsilon_{mnp}[\bar{\phi}^m, \bar{\phi}^n],\ [Q, f_{++}] = -i[\psi_{m+}, \bar{\phi}^m] \qquad Q^m_+ \in SU(1|3)$$

- index over BMN cohomologies: exhibits $O(N^2)$ entropy at large N

$$\frac{1}{N!}\int_0^{2\pi}\prod_{a=1}^{N}\frac{d\alpha_a}{2\pi}\cdot\prod_{a\neq b}\frac{(1-e^{i\alpha_{ab}})\prod_{I<J}(1-e^{-\Delta_I-\Delta_J}e^{i\alpha_{ab}})}{(1-e^{-\Delta_1-\Delta_2-\Delta_3}e^{i\alpha_{ab}})\prod_{I=1}^{3}(1-e^{-\Delta_I}e^{i\alpha_{ab}})}$$
$$\cdot\left[\frac{\prod_{I<J}(1-e^{-\Delta_I-\Delta_J})}{(1-e^{-\Delta_1-\Delta_2-\Delta_3})\prod_{I=1}^{3}(1-e^{-\Delta_I})}\right]^{N-1}$$

- unrefinement: $e^{-\Delta_1} = e^{-\Delta_2} = e^{-\Delta_3} \equiv t^2$ → conjugate charge $j \equiv 6(R+J)$, where $R \equiv \frac{R_1+R_2+R_3}{3}$, $J \equiv \frac{J_1+J_2}{2}$

BMN gravitons

- **Counting** independent finite N graviton cohomologies?

- All elements of $S_{n\geq N+1}$ can be decomposed into those of $S_{n\leq N}$ using Cayley-Hamilton equation.

- They are **absolutely protected**. → Go to **free limit** $g_{YM} \to 0$.

- Superconformal primaries $\mathrm{tr}[\bar{\phi}^{(m_1} \cdots \bar{\phi}^{m_n)}]$: scalars are symmetrized → Treat them as **diagonal matrices** containing **$N-1$ eigenvalues**.

- Superconformal descendants: SUSY transformation of diagonal fields only yields diagonal fields in the free theory. → Treat all elementary fields in $S_{n\leq N}$ to be **diagonal**. (All Q-actions vanish.)

- Conformal primaries in $S_{n\leq N}$ are now just the **Weyl-invariant polynomials of the eigenvalues**.

- Trace relations become now relations between **eigenvalue polynomials**.

ex) $SU(2)$: $\mathrm{tr}(X^2)\mathrm{tr}(Y^2) - [\mathrm{tr}(XY)]^2 \sim \mathrm{tr}([X,Y][X,Y]) \sim Q\mathrm{tr}(\psi_3[X,Y]) \longrightarrow x^2y^2 = (xy)^2$

$$X = \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix}, \dots$$

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 - Trace relations become now relations between **eigenvalue polynomials**.

- Counting independent **polynomials** ex) $\{p_i(x^2, y^2, xy)\}$ / $x^2y^2 = (xy)^2$

 - A well-known classic mathematical problem: the **Gröbner basis**

 - We could find a closed-form expression only for the $SU(2)$ theory

$$Z_{\mathrm{grav}}^{SU(2)} = \frac{1+3t^4-8t^6-6t^{10}+10t^{12}+9t^{14}-9t^{16}+16t^{18}-18t^{20}-3t^{22}+t^{24}-3t^{26}+9t^{28}-2t^{30}+3t^{32}-3t^{34}}{(1-t^4)^3(1-t^8)^3}$$

 - For $SU(3)$, only until $j \leq 54$, for $SU(4)$, only until $j \leq 30$

BMN gravitons

- We could find a closed-form expression only for the $SU(2)$ theory

$$Z_{\text{grav}}^{SU(2)} = \frac{1+3t^4-8t^6-6t^{10}+10t^{12}+9t^{14}-9t^{16}+16t^{18}-18t^{20}-3t^{22}+t^{24}-3t^{26}+9t^{28}-2t^{30}+3t^{32}-3t^{34}}{(1-t^4)^3(1-t^8)^3}$$

- For $SU(3)$, only until $j \leq 54$, for $SU(4)$, only until $j \leq 30$

$SU(2)$ BMN index

- Index over $SU(2)$ BMN cohomologies $\quad e^{-\Delta_1} = e^{-\Delta_2} = e^{-\Delta_3} \equiv t^2$

$$Z = \Big[1 + 3t^2 + 12t^4 + 20t^6 + 42t^8 + 48t^{10} + 75t^{12} + 66t^{14} + 81t^{16} + 55t^{18} + 54t^{20} + 27t^{22} + 19t^{24} + 6t^{26} + 3t^{28}\Big] \frac{(1-t^2)^3}{(1-t^{12})(1-t^8)^3}$$

- BMN single-trace gravitons in $SU(2)$: $\mathrm{tr}(\bar\phi^{(m}\bar\phi^{n)})$

$$\mathrm{tr}(\bar\phi^m\psi_{n+}) - \tfrac{1}{3}\delta^m_n \mathrm{tr}(\bar\phi^l\psi_{l+})$$
$$\mathrm{tr}(\bar\phi^m f_{++} - \tfrac{1}{4}\epsilon^{mnp}\psi_{n+}\psi_{p+})$$

- Index over $SU(2)$ **BMN black hole cohomologies**:

$$Z - Z_{\rm grav} = -\frac{e^{-4(\Delta_1+\Delta_2+\Delta_3)}}{1-e^{-2(\Delta_1+\Delta_2+\Delta_3)}} \cdot \prod_{I=1}^{3}(1-e^{-\Delta_I}) \cdot \prod_{I=1}^{3} \frac{1}{(1-e^{-\Delta_I}e^{-\Delta_1-\Delta_2-\Delta_3})}$$

$$Z - Z_{\rm grav} = -t^{24} + 3t^{26} - 3t^{28} + t^{30} - 3t^{32} + \cdots \, .$$

[Chang, Lin 22]

$SU(2)$ BMN black hole cohomologies

- Index over $SU(2)$ BMN black hole cohomologies:

$$Z - Z_{\text{grav}} = -\frac{e^{-4(\Delta_1+\Delta_2+\Delta_3)}}{1-e^{-2(\Delta_1+\Delta_2+\Delta_3)}} \cdot \prod_{I=1}^{3}(1-e^{-\Delta_I}) \cdot \prod_{I=1}^{3} \frac{1}{(1-e^{-\Delta_I}e^{-\Delta_1-\Delta_2-\Delta_3})}$$

- 1st factor: **'core' black hole primary** operators
- 2nd factor: $SU(1|3)$ descendants from Q_+^m action
- 3rd factor: multiplication of certain **multi-graviton states w_2** to the core black hole primaries

$$\mathbf{3} \quad : \quad xf - \tfrac{1}{2}\psi_2\psi_3 \ , \ yf - \tfrac{1}{2}\psi_3\psi_1 \ , \ zf - \tfrac{1}{2}\psi_1\psi_2 \ .$$

- Index of $SU(2)$ core black hole primaries: $-\dfrac{t^{24}}{1-t^{12}} = -t^{24} - t^{36} - t^{48} - t^{60} - \cdots$

→ **unique fermionic cohomology** at $j = 24 + 12n$.

$j \equiv 6(R+J)$, where $R \equiv \frac{R_1+R_2+R_3}{3}$, $J \equiv \frac{J_1+J_2}{2}$

- We shall use the 3-dimensional vector notation for $SU(2)$ adjoint fields $\phi^m = (X, Y, Z), \psi_m, f$

$$Q\phi^m = 0 \ , \quad Q\psi_m = \tfrac{1}{2}\epsilon_{mnp}\phi^n \times \phi^p \ , \ Qf = \phi^m \times \psi_m$$

$SU(2)$ threshold black hole cohomology

- O_0 at t^{24}: $O_0 \equiv \epsilon^{p_1p_2p_3} v^m{}_{p_1} v^n{}_{p_2} (\psi_m \cdot \psi_n \times \psi_{p_3})$ [SC, S.Kim, E.Lee, J.Park 22]

$$v^m{}_n \equiv (\phi^m \cdot \psi_n) - \frac{1}{3}\delta^m_n (\phi^p \cdot \psi_p)$$

- scaling dimension: $E = \frac{19}{2}$
- First two terms are the **graviton cohomologies**.
- Q-closed by explicit computation (**Fermi statistics**)

$$Q(\psi_m \cdot \psi_n \times \psi_p) = \frac{3}{2}\epsilon_{(m|qr}(\phi^q \times \phi^r)\cdot(\psi_{|n} \times \psi_{p)}) = 3\epsilon_{(m|qr}(\phi^q \cdot \psi_{|n})(\phi^r \cdot \psi_{p)}) = 3\epsilon_{(m|qr} v^q{}_{|n} v^r{}_{p)}$$

- not of graviton type since it contains seven (odd) letters
- not Q-exact since its descendant $Q^2_+ Q^1_+ O_0{}'$ contains nonzero $\phi^0\psi^7$ terms which cannot be made by Q-action

+) $(Z \cdot f + \psi_1 \cdot \psi_2) O_0'$ is not Q-exact since its descendant by $Q^2_+ Q^1_+$-action contains nonzero $\phi^0\psi^9$ terms.

- This operator **cannot be lifted** by any corrections, since there is no would-be pair operator.

$SU(2)$ core black hole primaries

- O_1 at t^{36}: $$\begin{aligned} O_1 &= (f\cdot f)\epsilon^{c_1c_2c_3}(\phi^a\cdot\psi_{c_1})(\phi^b\cdot\psi_{c_2})(\psi_a\cdot\psi_b\times\psi_{c_3}) \\ &+\epsilon^{b_1b_2b_3}\epsilon^{c_1c_2c_3}(f\cdot\psi_{b_1})(\phi^a\cdot\psi_{c_1})(\psi_{b_2}\cdot\psi_{c_2})(\psi_a\cdot\psi_{b_3}\times\psi_{c_3}) \\ &-\tfrac{1}{72}\epsilon^{a_1a_2a_3}\epsilon^{b_1b_2b_3}\epsilon^{c_1c_2c_3}(\psi_{a_1}\cdot\psi_{b_1}\times\psi_{c_1})(\psi_{a_2}\cdot\psi_{b_2}\times\psi_{c_2})(\psi_{a_3}\cdot\psi_{b_3}\times\psi_{c_3}) \end{aligned}$$
- O_n at t^{24+12n}: $O_n \equiv (f\cdot f)^n O_0 + n(f\cdot f)^{n-1} f\cdot\xi + \frac{2n^2+n}{3}(f\cdot f)^{n-1}\chi$ $$\begin{aligned} \vec{\xi} &= \epsilon^{b_1b_2b_3}\epsilon^{c_1c_2c_3}\vec{\psi}_{b_1}(\phi^a\cdot\psi_{c_1})(\psi_{b_2}\cdot\psi_{c_2})(\psi_a\cdot\psi_{b_3}\times\psi_{c_3}) \\ \chi &= -\tfrac{1}{72}\epsilon^{a_1a_2a_3}\epsilon^{b_1b_2b_3}\epsilon^{c_1c_2c_3}(\psi_{a_1}\cdot\psi_{b_1}\times\psi_{c_1})(\psi_{a_2}\cdot\psi_{b_2}\times\psi_{c_2})(\psi_{a_3}\cdot\psi_{b_3}\times\psi_{c_3}) \\ &= -120\psi_1^1\psi_1^2\psi_1^3\psi_2^1\psi_2^2\psi_2^3\psi_3^1\psi_3^2\psi_3^3\ . \end{aligned}$$

- $O_m O_n = 0$ by Fermi statistics (all terms have at least 10 fermions)

→ These account for the $SU(2)$ BMN index.

Black hole from cohomology?

- O_0 at t^{24}: $O_0 \equiv \epsilon^{p_1 p_2 p_3} v^m{}_{p_1} v^n{}_{p_2} (\psi_m \cdot \psi_n \times \psi_{p_3})$ $\qquad v^m{}_n \equiv (\phi^m \cdot \psi_n) - \frac{1}{3}\delta^m_n (\phi^p \cdot \psi_p)$

  O_n at t^{24+12n}: $O_n \equiv (f \cdot f)^n O_0 + n(f \cdot f)^{n-1} f \cdot \xi + \frac{2n^2+n}{3}(f \cdot f)^{n-1}\chi$ $\qquad O_m O_n = 0$

  - **giant graviton** interpretation? [Imamura 21, 22] [Gaiotto, J.H.Lee 21, 22] [Murthy 22] [SC, S.Kim, E.Lee, J.Lee 22]
  - **Fermi surface** model? [Berkooz, Reichmann, Simon 06]
  - a **tower** structure with increasing angular momentum ~ a higher-spin tower protected by symmetries?

- Can we understand some properties of these 'black hole cohomologies' from the **semi-classical black holes** in AdS?
  - **geometrical** interpretation? → finite energy **excitation outside the horizon**
  - $SU(2)$ vs. $SU(\infty)$

No-hair theorem

- O_0 abhors the dressing by certain gravitons, reminiscent of the black hole **no-hair theorem**.

 - Three of them are explicitly shown to be ***Q*-exact**.

$$
\begin{aligned}
O_0(\bar\phi^{(m}\cdot\bar\phi^{n)}) = -\frac{1}{14}Q[&20\epsilon^{rs(m}(\bar\phi^{n)}\cdot\psi_{p+})(\bar\phi^{p}\cdot\psi_{r+})(\bar\phi^{q}\cdot\psi_{q+})(f_{++}\cdot\psi_{s+})\\
&-20\epsilon^{prs}(\bar\phi^{(m}\cdot\psi_{p+})(\bar\phi^{n)}\cdot\psi_{r+})(\bar\phi^{q}\cdot\psi_{q+})(f_{++}\cdot\psi_{s+})\\
&+30\epsilon^{prs}(\bar\phi^{(m}\cdot\psi_{p+})(\bar\phi^{n)}\cdot\psi_{r+})(\bar\phi^{q}\cdot\psi_{s+})(f_{++}\cdot\psi_{q+})\\
&-7\epsilon^{a_1a_2p}\epsilon^{b_1b_2(m}(\bar\phi^{n)}\cdot\psi_{p+})(\bar\phi^{q}\cdot\psi_{q+})(\psi_{a_1+}\cdot\psi_{a_2+})(\psi_{b_1+}\cdot\psi_{b_2+})\\
&+18\epsilon^{a_1a_2p}\epsilon^{b_1b_2(m}(\bar\phi^{n)}\cdot\psi_{q+})(\bar\phi^{q}\cdot\psi_{p+})(\psi_{a_1+}\cdot\psi_{a_2+})(\psi_{b_1+}\cdot\psi_{b_2+})]
\end{aligned}
$$

$$
\begin{aligned}
O_0(\bar\phi^{m}\cdot\bar\lambda_{\dot\alpha}) = \frac{1}{8}Q[&40\epsilon^{mnp}(f_{++}\cdot\psi_{q+})(\bar\lambda_{\dot\alpha}\cdot\psi_{r+})(\bar\phi^{q}\cdot\psi_{n+})(\bar\phi^{r}\cdot\psi_{p+})\\
&-4\epsilon^{ma_1a_2}\epsilon^{nb_1b_2}(\bar\lambda_{\dot\alpha}\cdot\psi_{n+})(\bar\phi^{p}\cdot\psi_{p+})(\psi_{a_1+}\cdot\psi_{a_2+})(\psi_{b_1+}\cdot\psi_{b_2+})\\
&+6\epsilon^{ma_1a_2}\epsilon^{nb_1b_2}(\bar\lambda_{\dot\alpha}\cdot\psi_{p+})(\bar\phi^{p}\cdot\psi_{n+})(\psi_{a_1+}\cdot\psi_{a_2+})(\psi_{b_1+}\cdot\psi_{b_2+})\\
&+\epsilon^{na_1a_2}\epsilon^{pb_1b_2}(\bar\lambda_{\dot\alpha}\cdot\psi_{n+})(\bar\phi^{m}\cdot\psi_{p+})(\psi_{a_1+}\cdot\psi_{a_2+})(\psi_{b_1+}\cdot\psi_{b_2+})]\,.
\end{aligned}
$$

$$
\begin{aligned}
&O_0\left(\bar\phi^{m}\cdot\psi_{n+}-\tfrac{1}{3}\delta^m_n\bar\phi^{p}\cdot\psi_{p+}\right)\\
&=\frac{1}{4}Q\left[\epsilon_{npq}\epsilon^{ra_1a_2}\epsilon^{qb_1b_2}\epsilon^{mc_1c_2}(\bar\phi^{p}\cdot\psi_{r+})(\psi_{a_1+}\cdot\psi_{a_2+})(\psi_{b_1+}\cdot\psi_{b_2+})(\psi_{c_1+}\cdot\psi_{c_2+})\right]
\end{aligned}
$$

General black hole cohomologies

- As charge order increases, it becomes more and more difficult to judge the Q-exactness of possible product operators.

- We shall try to see whether the index can be explained by the **products** of known gravitons and core primaries O_n, together with their descendants, without introducing any new core black hole primaries.

- Index over **black hole cohomologies** in general sector:

$$\begin{aligned} Z - Z_{\text{grav}} \;&=\; \big[-t^{24} - \chi_{(1,3)}t^{32} - (\chi_{(1,\bar{3})} + \chi_{(3,6)})t^{34} - \chi_{(2,3)}t^{35} + (\chi_{(3,1)} + \chi_{(3,8)})t^{36} \\ &\quad -(\chi_{(2,\bar{3})} + \chi_{(4,6)})t^{37} + \chi_{(5,3)}t^{38} + (\chi_{(2,1)} + 2\chi_{(4,1)} + \chi_{(4,8)})t^{39} \\ &\quad -(2\chi_{(1,6)} + \chi_{(3,\bar{3})} + \chi_{(5,\bar{3})} + \chi_{(5,6)})t^{40}\big]\,\chi_{\text{D}} + \mathcal{O}(t^{41}) \;. \end{aligned}$$

$$\chi_{(2J'+1,\text{R})} \equiv \chi_{J'}^{SU(2)_R}(p)\chi_{\text{R}}^{SU(3)}(x,y) \tag{3.55}$$

$$\chi_D \equiv \frac{(1-t^2z_1)(1-\frac{t^2}{z_2})(1-\frac{t^2z_2}{z_1})\cdot(1-\frac{tp}{z_1})(1-\frac{t}{pz_1})(1-tz_2p)(1-\frac{tz_2}{p})(1-\frac{tz_1p}{z_2})(1-\frac{tz_1}{z_2p})}{(1-t^3p)(1-\frac{t^3}{p})}$$

$$t^6 = e^{-\Delta_1-\Delta_2-\Delta_3} = e^{-\omega_1-\omega_2},\;\; z_1 = e^{\frac{-2\Delta_1+\Delta_2+\Delta_3}{3}},\;\; z_2^{-1} = e^{\frac{\Delta_1-2\Delta_2+\Delta_3}{3}},\;\; p = e^{\frac{-\omega_1+\omega_2}{2}}$$

Partial no-hair theorem

- Index **does not capture any new** black hole cohomologies in $t^{25} \sim t^{31}$ except for the descendants of O_0.

 - Possible independent cohomologies in that range:

$$O_0(\bar{\phi}^{(m} \cdot \bar{\phi}^{n)}), \quad O_0(\bar{\phi}^m \cdot \bar{\lambda}_{\dot{\alpha}}) , \quad O_0(\bar{\lambda}_{\dot{+}} \cdot \bar{\lambda}_{\dot{-}}) ,$$

$$O_0(\bar{\phi}^m \cdot \psi_{n+} - \tfrac{1}{3}\delta^m_n \bar{\phi}^p \cdot \psi_{p+}) , \quad O_0(\bar{\lambda}_{\dot{\alpha}} \cdot \psi_{m+} - \tfrac{1}{2}\epsilon_{mnp}\phi^n \cdot D_{+\dot{\alpha}}\phi^p) ,$$

$$O_0 \partial_{+\dot{\alpha}}(\bar{\phi}^{(m} \cdot \bar{\phi}^{n)}) .$$

 - Three of them are explicitly shown to be ***Q*-exact** and the others may also be Q-exact.

 → These Q-exactness implies that O_0 abhors the dressing by certain gravitons, reminiscent of the black hole **no-hair theorem**.

- $-\chi_{(1,3)} t^{32}$: $\quad O_0(\bar{\phi}^m \cdot f_{++} + \tfrac{1}{2}\epsilon^{mnp}\psi_{n+} \cdot \psi_{p+})$

 - No-hair interpretation holds only for certain low-lying gravitons.

 - Out of 32 particle species of conformal primaries in S_2, 29 gravitons except the above gravitons do not appear in the index when they are multiplied to O_0.

 - Similar theorem for all O_n in the BMN sector.

 → **Partial no-hair theorem**: 'allowed' hairs?

Hairy black hole primaries

- $0t^{33}$: $O_0\partial_{+\dot\alpha}(\bar\lambda_{\dot\beta}\cdot\bar\lambda^{\dot\beta})$, $O_0\left(f_{++}\cdot\bar\lambda_{\dot\alpha}+\frac{2}{3}\psi_{m+}\cdot D_{+\dot\alpha}\bar\phi^m-\frac{1}{3}\bar\phi^m\cdot D_{+\dot\alpha}\psi_{m+}\right)$
- $-(\chi_{(1,\bar 3)}+\chi_{(3,6)})t^{34}$: $O_0\partial_+{}^{\dot\alpha}\left(\bar\lambda_{\dot\alpha}\cdot\psi_{m+}-\frac{1}{2}\epsilon_{mnp}\bar\phi^n\cdot D_{+\dot\alpha}\bar\phi^p\right)$, $O_0\partial_{+\dot\alpha}\partial_{+\dot\beta}(\bar\phi^m\cdot\bar\phi^n)$
- $-\chi_{(2,3)}t^{35}$: $O_0\partial_{+\dot\alpha}(f_{++}\cdot\bar\phi^m+\frac{1}{2}\epsilon^{mnp}\psi_{n+}\cdot\psi_{p+})$
- $(-1+1+\chi_{(3,1)}+\chi_{(3,8)})t^{36}$: O_1, $O_0\partial_{+\dot\alpha}(f_{++}\cdot\bar\lambda_{\dot\beta}+\frac{2}{3}\psi_{m+}\cdot D_{+\dot\beta}\bar\phi^m-\frac{1}{3}\bar\phi^m\cdot D_{+\dot\beta}\psi_{m+})$, $O_0\partial_{+\dot\alpha}\partial_{+\dot\beta}(\bar\phi^m\cdot\psi_{n+}-\frac{1}{3}\delta^m_n\bar\phi^p\cdot\psi_{p+})$
- $-(\chi_{(2,\bar 3)}+\chi_{(4,6)})t^{37}$: $O_0\partial_{+\dot\alpha}\partial_+{}^{\dot\beta}(\bar\lambda_{\dot\beta}\cdot\psi_{m+}-\frac{1}{2}\epsilon_{mnp}\bar\phi^n\cdot D_{+\dot\beta}\bar\phi^p)$, $O_0\partial_{+\dot\alpha}\partial_{+\dot\beta}\partial_{+\dot\gamma}(\bar\phi^m\cdot\bar\phi^n)$
- $\chi_{(5,3)}t^{38}$: $O_0\partial_{+(\dot\alpha}\partial_{+\dot\beta}\partial_{+\dot\gamma}(\bar\lambda_{\dot\delta)}\cdot\bar\phi^m)$

  $O_0\partial_{+\dot\alpha}\partial_{+\dot\beta}\partial_+{}^{\dot\gamma}(\bar\lambda_{\dot\gamma}\cdot\bar\phi^m)$, $O_0\partial_{+\dot\alpha}\partial_{+\dot\beta}(f_{++}\cdot\bar\phi^m+\frac{1}{2}\epsilon^{mnp}\psi_{n+}\cdot\psi_{p+})$

New core black hole primaries

- $(\chi_{(2,1)} + 2\chi_{(4,1)} + \chi_{(4,8)})t^{39}$

  - Possible product cohomologies:

$$
\begin{aligned}
(\mathbf{4},\mathbf{1})^{\mathrm{F}} &: \; O_0 \partial_{+\dot\alpha}\partial_{+\dot\beta}\partial_{+\dot\gamma}(\bar\lambda_{\dot\delta}\cdot\bar\lambda^{\dot\delta})\,, \\
(\mathbf{2},\mathbf{1})^{\mathrm{B}} &: \; O_0 \partial_{+\dot\alpha}\partial_{+}{}^{\dot\beta}(f_{++}\cdot\bar\lambda_{\dot\beta} + \tfrac{2}{3}\psi_{m+}\cdot D_{+\dot\beta}\bar\phi^m - \tfrac{1}{3}\bar\phi^m\cdot D_{+\dot\beta}\psi_{m+})\,, \\
(\mathbf{4},\mathbf{1})^{\mathrm{B}} &: \; O_0 \partial_{+(\dot\alpha}\partial_{+\dot\beta}(f_{++}\cdot\bar\lambda_{\dot\gamma)} + \tfrac{2}{3}\psi_{m+}\cdot D_{+\dot\gamma)}\bar\phi^m - \tfrac{1}{3}\bar\phi^m\cdot D_{+\dot\gamma)}\bar\psi_m)\,, \\
(\mathbf{4},\mathbf{8})^{\mathrm{B}} &: \; O_0 \partial_{+\dot\alpha}\partial_{+\dot\beta}\partial_{+\dot\gamma}(\bar\phi^m\cdot\psi_{n+} - \tfrac{1}{3}\delta^m_n\bar\phi^p\cdot\psi_{p+})\,.
\end{aligned}
$$

→ At least **4 core black hole primaries** are needed to account for $\chi_{(4,1)}t^{39}$. (upper bound)

- Index admits natural explanation in terms of **hairy black hole (grey galaxy) cohomologies** in a wide range $t^{33}\sim t^{38}$.

  - Most of the graviton hairs appeared in this range are **conformal descendants** in S_2 .

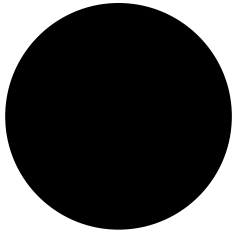
  → Can we find similar behavior from the semiclassical black holes in AdS?

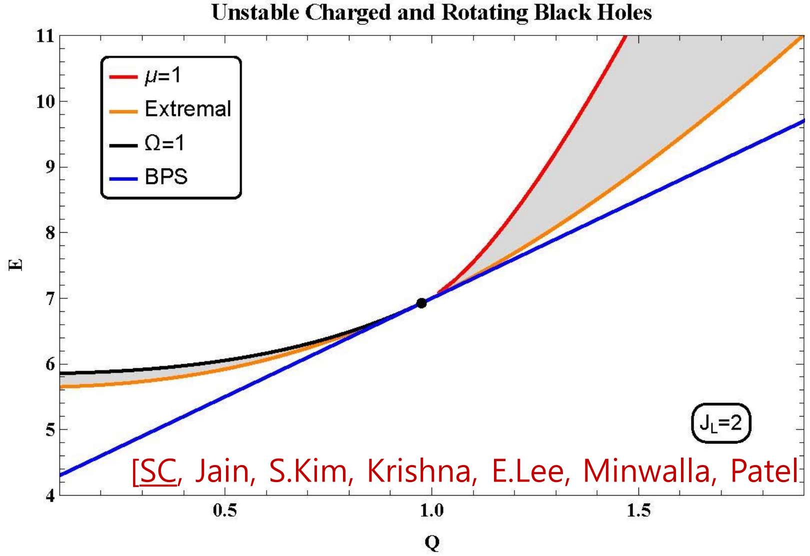
Particle on BPS black holes in AdS_5

- Simplest BPS black holes in AdS_5 are parametrized by **one free parameter** q. [Gutowski, Reall 04]

$$M = \frac{3R+2J}{\ell} \quad , \quad R = \frac{N^2}{2}\left(\frac{q}{\ell^2} + \frac{q^2}{2\ell^2}\right) \quad , \quad J = \frac{N^2}{2}\left(\frac{3q^2}{2\ell^4} + \frac{q^3}{\ell^6}\right)$$

$$\mathcal{G}(R,J) \equiv R^3 + \frac{N^2}{2}J^2 - \left(3R + \frac{N^2}{2}\right)(3R^2 - N^2 J) = 0$$





[SC, Jain, S.Kim, Krishna, E.Lee, Minwalla, Patel 24]

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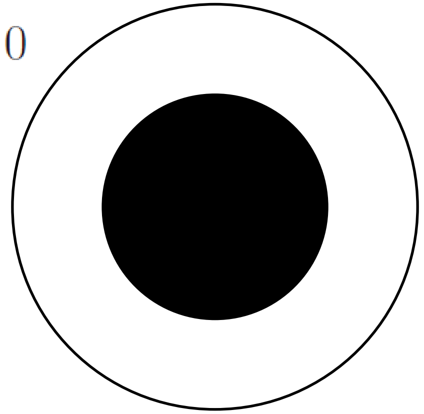
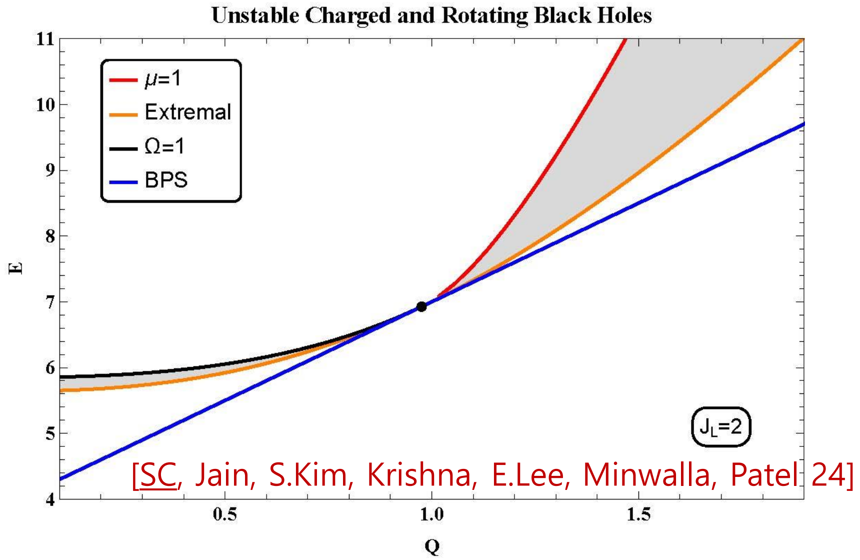




[SC, Jain, S.Kim, Krishna, E.Lee, Minwalla, Patel 24]

- Bosonic **probe particle**, dual to the chiral primary $|n\rangle$, in the black hole background:

$$(M\ell)^2 = n(n-4) \approx n^2 \, , \quad \delta R = \frac{n}{3} \qquad S = \frac{n}{\ell}\int d\tau \left[-\sqrt{-g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu} - \frac{1}{3}A_\mu \dot{x}^\mu\right]$$

- rotating BPS solutions **outside the horizon**

$$x = \ell^2\left(\frac{\delta J_1 + \delta J_2}{3\delta R} - \frac{2q}{\ell^2}\right) \equiv \ell^2\left(\frac{2\delta J}{3\delta R} - \frac{2q}{\ell^2}\right) > 0 \;\rightarrow\; \delta J > \frac{nq}{\ell^2} \, . \qquad \delta\mathcal{G}(R,J) \approx \frac{N^4}{2}\left(1+\frac{q}{\ell^2}\right)^3\left(\delta J - \frac{3q}{\ell^2}\delta R\right)$$

Hairy BPS black holes in AdS_5

- '**Non-interacting mix**' of small black holes and dilute graviton hairs

  - superposition of non-hairy black hole and the graviton wavefunction in vacuum AdS

[Basu, Bhattacharya, Bhattacharyya, Loganayagam, Minwalla, Umesh 10]
[Bhattacharya, Minwalla, Papadodimas 10]

- We study a **small complex scalar** Φ in the hypermultiplet on the BPS black hole background.

  - The solutions to the BPS equations: $\Phi(x,\theta,\phi,\psi) = \Phi(x)e^{im\psi}f(\theta,\phi)$

[Bellorin, Ortin 07]
[Liu, Lu, Pope, Vazquez-Poritz 07]

$$\Phi(x) = \varepsilon x^{\frac{m-2q/\ell^2}{1+3q/\ell^2}}\left(1+\frac{3q}{\ell^2}+\frac{x}{\ell^2}\right)^{-\frac{1+m+q/\ell^2}{1+3q/\ell^2}} \quad f(\theta,\phi)e^{im\psi} = (\cos\tfrac{\theta}{2}e^{i\phi_1})^{m_1}(\sin\tfrac{\theta}{2}e^{i\phi_2})^{m_2} \quad , \quad (\psi,\phi) = \phi_1 \pm \phi_2$$

$$m_1 + m_2 = 2m \text{ and } m_1, m_2 = 0,1,2,\cdots$$

  - The angle-dependent part is the usual '**derivative**' factor $(\partial_{++})^{m_1}(\partial_{+-})^{m_2}$ acting on **conformal primaries** in dual QFT.

  - At the horizon $x \to 0$, the solution remains **finite** only for modes satisfying $m = \dfrac{m_1+m_2}{2} \geq \dfrac{2q}{\ell^2}$

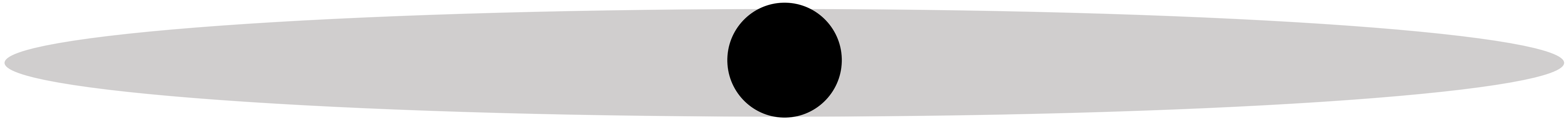
  - (A special case of) BPS **grey galaxies** [S.Kim, Kundu, E.Lee, J.Lee, Minwalla 23] [Bajaj, Kumar, Minwalla, Mukherjee, Rahaman 24] [SC, Jain, S.Kim, Krishna, G.Kwon, E.Lee, Minwalla 25]

Well-behaved BPS hairs

- If the quantum number m is **large**, then the averaged radial position of fields in vacuum AdS is **far away from the center**.

 - **Conformal primary** with $m = 0$ prefers to be at the **central** region of AdS.
 - In the black hole background, the central region is hidden inside the **event horizon**.
 - Fields with lower m will be more likely to be swallowed by the black hole and back-react more **strongly**.
 - Fields with higher m mostly rotates **outside** the horizon and can be superposed with **small** back-reaction.

→ Thermal ensemble: BPS grey galaxies [S.Kim, Kundu, E.Lee, J.Lee, Minwalla 23] [Bajaj, Kumar, Minwalla, Mukherjee, Rahaman 24] [SC, Jain, S.Kim, Krishna, G.Kwon, E.Lee, Minwalla 25]



- We found why **conformal primary** states of gravitons **cannot** be the hairs of BPS black holes in superposition picture.
- We also found that particles with **large orbital angular momentum** can provide hairs of BPS black holes.

→ These are in qualitative accordance with the behavior of the **CFT operator spectrum** of $SU(2)$ theory.

Instabilities of near-extremal BH

- All **(near-)extremal BH** except for the BPS black holes are **thermodynamically unstable** due to $\nu > 1$.
  - The first law of black hole thermodynamics: $dS = \beta(dE - \Omega_i dJ_i - \Delta_I dQ_I)$
  - The BPS bound: $E \geq J_1 + J_2 + Q_1 + Q_2 + Q_3$
  - Consider a process of emitting BPS particles from the BH: $dS = (1 - \Omega_i)\beta dJ_i + (1 - \Delta_I)\beta dQ_I > 0$

  → **emitting charges** conjugate to chemical potentials with $\nu > 1$ will increase the entropy !

  : **thermodynamically more favorable** to emit charges to decrease those of BH

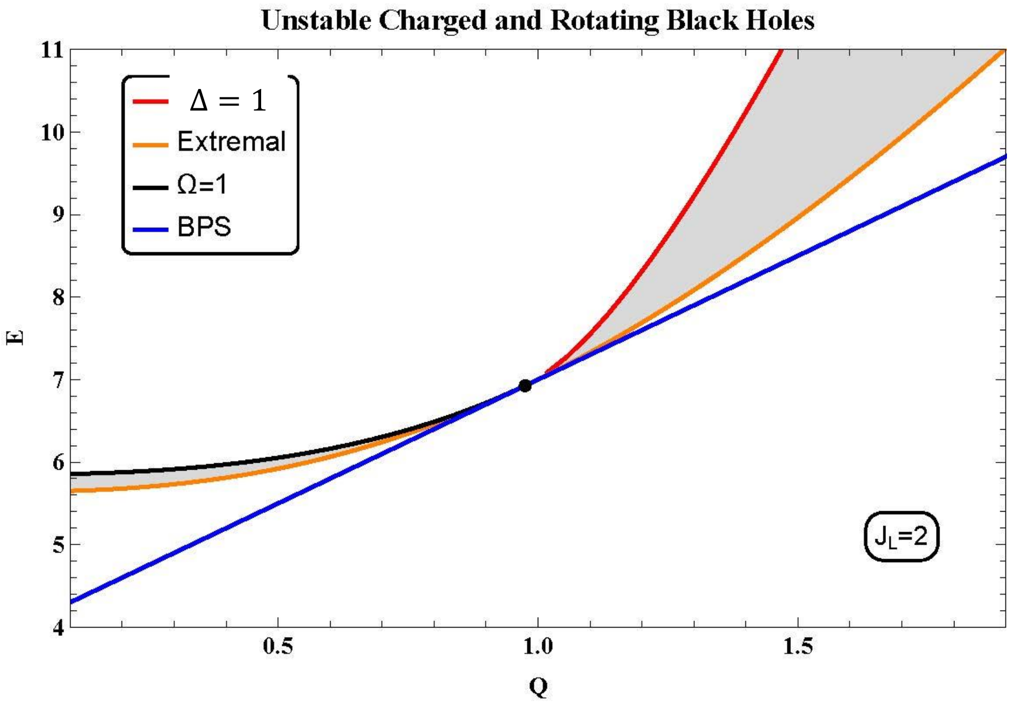
- In fact, they suffer from dynamic instabilities

  : **superradiant instabilities**

  → should **decay into stable** configurations

1) Construct black hole solutions **stable** under superradiant instabilities.

2) **Smoothly extend** such solutions to the BPS surface !



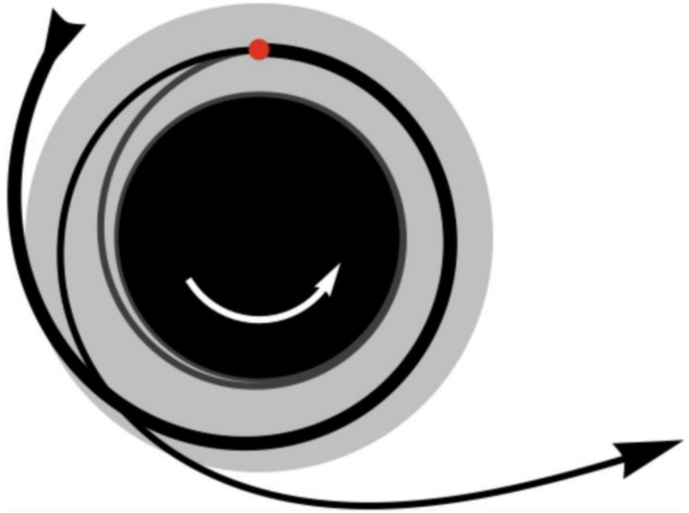
Kerr-AdS superradiant instability

- **Over-spinning Kerr-AdS black holes** in $D \geq 4$ suffer from classical super-radiant instabilities. [Cardoso, Dias 04]

  - repeated sequences of the **super-radiance** and the **reflection** by the AdS wall

  → **Tachyonic quasi-normal modes**

  ~ **emitting rotating hairs** to decay



Kerr-AdS superradiant instability → Grey Galaxy

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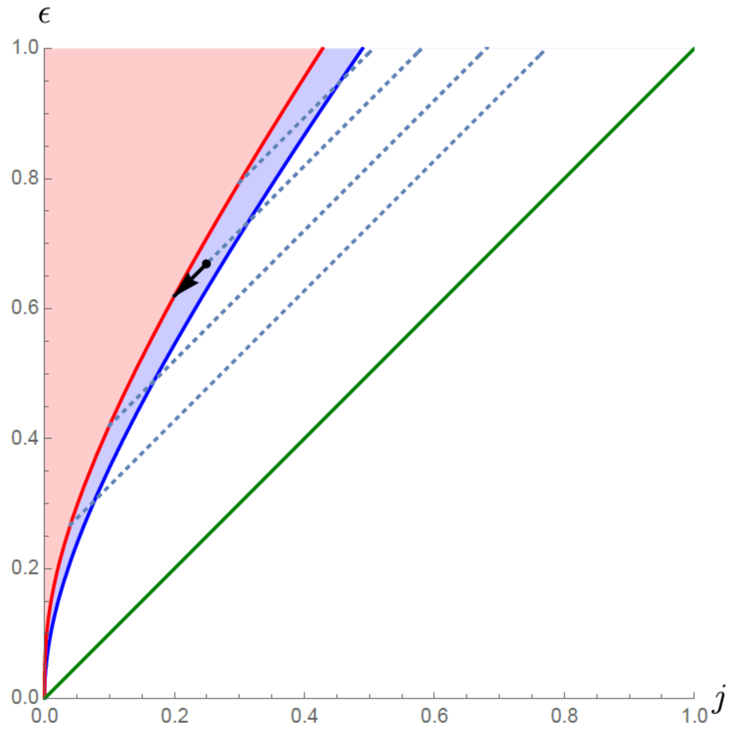


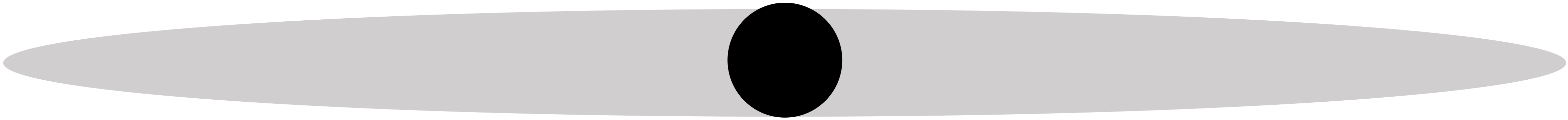
- **Grey Galaxies**: [S.Kim, Kundu, E.Lee, J.Lee, Minwalla, Patel 24]

  a **core black hole** with $\Omega = 1$ at the center of AdS,

  surrounded by a very **large** ($\sim N^{1/2}$) flat **revolving** disk of

  dilute thermal gas of gravitons ($E_{gas} \sim J_{gas} \sim N^2$)

  ~ **weakly-interacting mix** between BH and graviton gas



Charged-AdS superradiant instability

- **Over-charged RN-AdS black holes** in $AdS_5 \times S^5$ suffer from classical super-radiant instabilities. [Gubser 08]

  - repeated sequences of the **super-radiance** and the **reflection** by the AdS wall

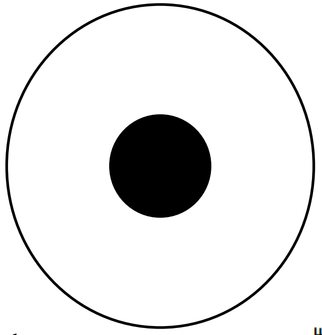
  → **Tachyonic quasi-normal modes**

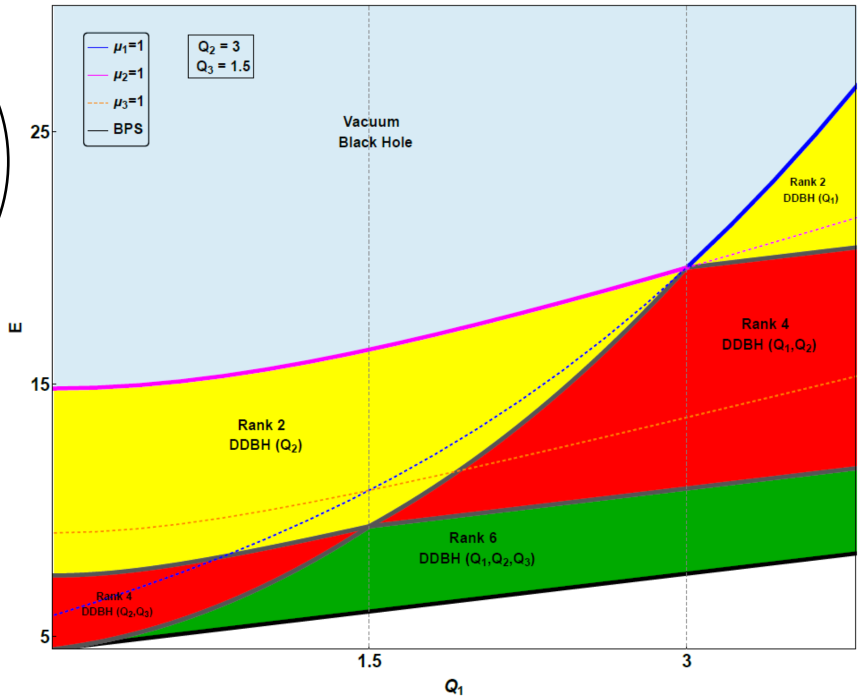
  ~ **emitting charged hairs** to decay

Charged-AdS superradiant instability → DDBH

- **Over-charged RN-AdS black holes** in $AdS_5 \times S^5$ suffer from classical super-radiant instabilities. [Gubser 08]
  - repeated sequences of the **super-radiance** and the **reflection** by the AdS wall
  - → **Tachyonic quasi-normal modes**
  - ~ **emitting charged hairs** to decay

- **Dual Dressed Black Holes** (DDBHs): [SC, Jain, S.Kim, Krishna, E.Lee, Minwalla, Patel 24]
  - a **core black hole** with $\Delta = 1$ at the center of AdS,
  - surrounded by a very **large** $(\sim N^{1/2})$
  - **dual giant** gravitons $(E_{dual} \sim Q_{dual} \sim N^2)$





  - ~ **weakly-interacting mix** between BH and dual giants

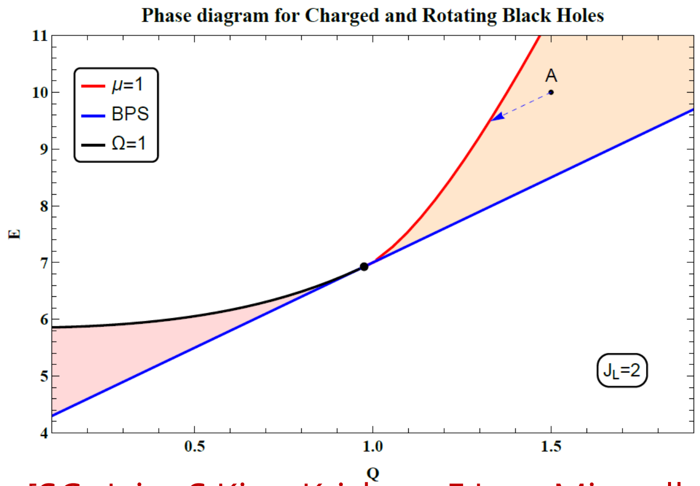
- DDBHs are the solution of full **10d string theory** and cannot be written down within a 5d supergravity with a finite number of fields.

Supersymmetric grey galaxies and DDBHs

- **BPS black holes** in $AdS_5 \times S^5$ carry 6 charges satisfying the BPS relation: $E = Q_1 + Q_2 + Q_3 + J_1 + J_2$
  - They satisfy another **nonlinear relation** to avoid CTCs:

$$\frac{j_1 j_2}{2} + q_1 q_2 q_3 = \left(q_1 + q_2 + q_3 + \frac{1}{2}\right)\left(q_1 q_2 + q_1 q_3 + q_2 q_3 - \frac{1}{2}(j_1 + j_2)\right)$$

- Replace the core black holes to the **BPS black holes**.
  → **Supersymmetric grey galaxies and DDBHs**

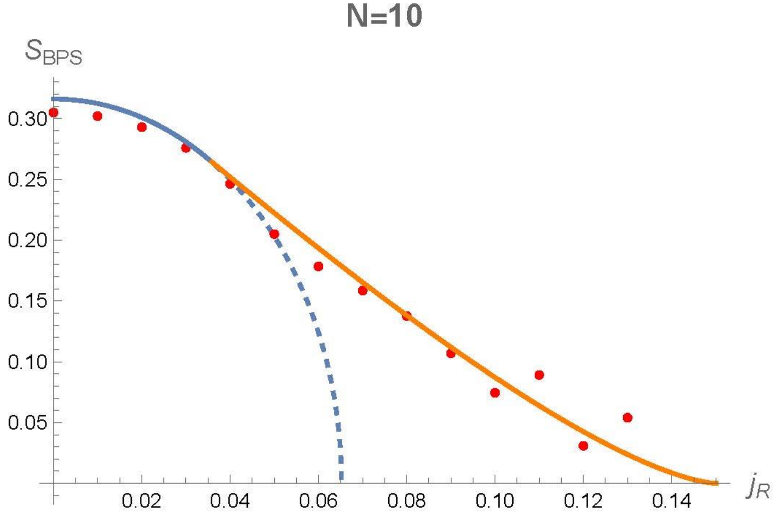


[SC, Jain, S.Kim, Krishna, E.Lee, Minwalla, Patel 24]

- Graviton gas and dual giants need not to obey the charge relation.
  → Those hairy BPS BHs can **freely violate the charge relation**.

- Hairy BHs can give more **dominant** contributions to the index.
  - They exist even when the non-hairy BHs **do not exist**.
  - Qualitative **agreement** with the numerically computed **index** at $N = 10$
  - Dominance in the microcanonical ensemble changes precisely when Kontsevich-Segal-Witten criterion for the unhiary BH is violated.

[Genolini, Janssen, Murthy 26; Krishna, Larsen 26]



[SC, Jain, S.Kim, Krishna, G.Kwon, E.Lee, Minwalla, Patel 25]

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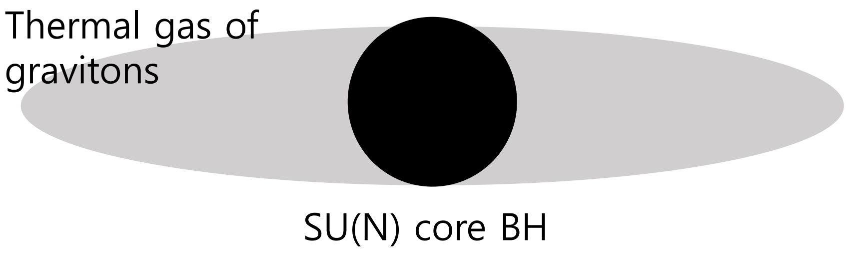
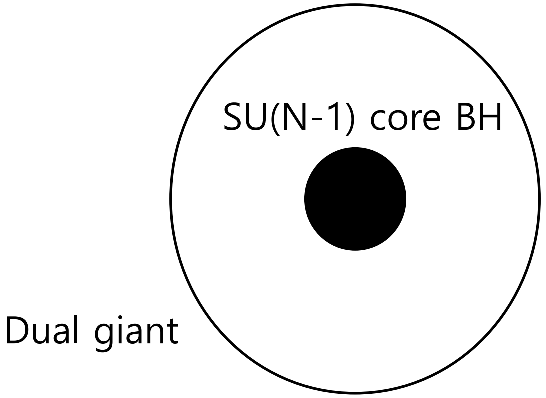
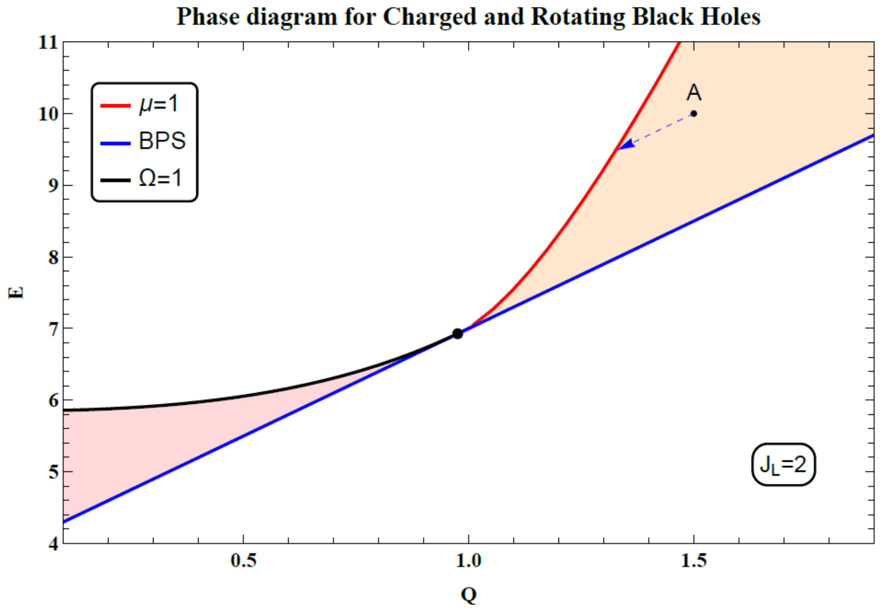






[SC, Jain, S.Kim, Krishna, E.Lee, Minwalla, Patel 24]

$SU(3)$ BMN black hole index

- BMN single-trace gravitons in $SU(3)$

$$
\begin{aligned}
u^{ij} &\equiv \mathrm{tr}\left(\phi^{(i}\phi^{j)}\right) \;, \quad u^{ijk} \equiv \mathrm{tr}\left(\phi^{(i}\phi^{j}\phi^{k)}\right) \;, \\
v^{i}{}_{j} &\equiv \mathrm{tr}\left(\phi^{i}\psi_{j}\right) - \tfrac{1}{3}\delta^{i}_{j}\mathrm{tr}\left(\phi^{a}\psi_{a}\right) \;, \quad v^{ij}{}_{k} \equiv \mathrm{tr}\left(\phi^{(i}\phi^{j)}\psi_{k}\right) - \tfrac{1}{4}\delta^{i}_{k}\mathrm{tr}\left(\phi^{(j}\phi^{a)}\psi_{a}\right) - \tfrac{1}{4}\delta^{j}_{k}\mathrm{tr}\left(\phi^{(i}\phi^{a)}\psi_{a}\right) \;, \\
w^{i} &\equiv \mathrm{tr}\left(f\phi^{i} + \tfrac{1}{2}\epsilon^{ia_1a_2}\psi_{a_1}\psi_{a_2}\right) \;, \quad w^{ij} \equiv \mathrm{tr}\left(f\phi^{(i}\phi^{j)} + \epsilon^{a_1a_2(i}\phi^{j)}\psi_{a_1}\psi_{a_2}\right) \;,
\end{aligned}
$$

- Index over $SU(3)$ **BMN black hole cohomologies**:

$$
Z - Z_{\mathrm{grav}} = Z_{\mathrm{core}}(\Delta_I) \cdot \prod_{I=1}^{3} \frac{1}{1 - e^{-\Delta_I} e^{-\Delta_1-\Delta_2-\Delta_3}} \cdot \prod_{I<J}\left(1 - e^{-\Delta_I-\Delta_J}\right)
$$

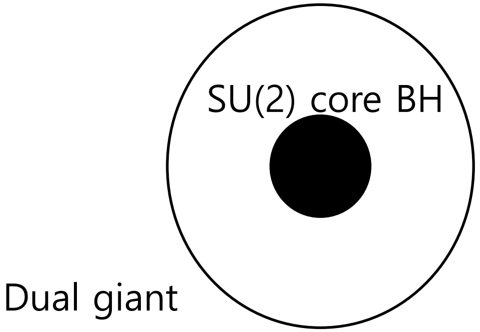
- Index over core primaries: $Z_{\mathrm{core}}(\Delta_I) \equiv f(t,x,y)$ with $e^{-\Delta_1} = t^2 x$, $e^{-\Delta_2} = t^2 y^{-1}$, $e^{-\Delta_3} = t^2 x^{-1} y$

$$
f(t,x,y) = \sum_{j=0}^{54} \sum_{\mathbf{R}_j} (-1)^{F(\mathbf{R}_j)} \chi_{\mathbf{R}_j}(x,y) t^j + \mathcal{O}(t^{56})
$$

$SU(3)$ core primaries

- Index over core primaries: $f(t,x,y) = \sum_{j=0}^{54} \sum_{\mathbf{R}_j} (-1)^{F(\mathbf{R}_j)} \chi_{\mathbf{R}_j}(x,y) t^j + \mathcal{O}(t^{56})$
- Partial **no hair** behavior: no $u_{2,3}, v_{2,3}$ hairs
- F_1 tower: **Kaluza-Klein tower** with increasing number of scalars
  ~ BPS **Dual Dressed BHs**:
  BH surrounded by a large dual giant graviton (D3-brane)

[SC, Jain, S.Kim, Krishna, E.Lee, Minwalla, Patel 24]
[de Mello Koch, M.Kim, S.Kim, J.Lee, S.Lee 24]
[SC, Jain, S.Kim, Krishna, G.Kwon, E.Lee, Minwalla 25]



cf) **higher spin** tower of $SU(2)$ ~ F_0: $tr(f^2), tr(f^3)$

[Gaikwad, Kibe, van Leuven, Mathieson 25]

j	F_0	F_1	F_2	F_3	F_4	F_{exc}	B_1	B_2	B_3	B_{exc}
24	[0, 0]									
26										
28										
30	[0, 0]	[3, 0]								
32		[4, 0]								
34		[5, 0]					[3, 1]			
36	[0, 0]	[6, 0]					[4, 1]			[3, 0]
38		[7, 0]				[1, 0]	[5, 1]			
40		[8, 0]	[5, 0]		[3, 1]		[6, 1]			
42	[0, 0]	[9, 0]	[6, 0]		[4, 1]		[7, 1]			[1, 1]
44		[10, 0]	[7, 0]		[5, 1]		[8, 1]	[5, 1]		
46		[11, 0]	[8, 0]		[6, 1]	[2, 0]	[9, 1]	[6, 1]		[5, 0]
48	[0, 0]	[12, 0]	[9, 0]		[7, 1]	[3, 0]	[10, 1]	[7, 1]		[4, 1]
50		[13, 0]	[10, 0]	[7, 0]	[8, 1]		[11, 1]	[8, 1]		[4, 0]
52		[14,0]	[11,0]	[8,0]	[9,1]	[2,0]	[12, 1]	[9,1]		[3,1]
54		[15,0]	[12,0]	[9,0]	[10,1]	[4,1]	[13, 1]	[10,1]	[7,1]	

Black hole ansatz

- Construct black hole cohomologies based on an **ansatz**.
  - Q-closedness should be ensured by the **trace relations** of $SU(N)$ matrices
  - We do not know the complete set of the trace relations for $N > 2$
  - However, we already know many of them for $SU(3)$: the **trace relations between gravitons**

- $SU(2)$ threshold cohomology: $O_0 \equiv \epsilon^{abc}(v_2)^m{}_a(v_2)^n{}_b \mathrm{tr}(\psi_{(c}\psi_m\psi_{n)})$
  - Q -action: $Q\mathrm{tr}(\psi_{(c}\psi_m\psi_{n)}) \propto \epsilon_{ab(c}(v_2)^a{}_m(v_2)^b{}_{n)} \equiv R(v_2)_{cmn}$

$$QO_0 \propto \epsilon^{abc}(v_2)^m{}_a(v_2)^n{}_b R(v_2)_{cmn} = 0 \ .$$

  - The first step is the ***SU*(2) graviton trace relation**.
  - At the last step, one can show that the quartic mesonic polynomial is identically zero.

→ A **relation of (trace) relations**.

Black hole ansatz

- **Trace relation**: the linear dependence between the multi-trace operators, up to Q-exact operators, due to the finite size of the matrices
  - They can be seen at the level of **gluons** ϕ, ψ, f.

$$R_a = \sum_i c_a\, g_i(\phi, \psi, f) = Q r_a$$

- **Relation of (trace) relations**: a linear combination of the trace relations, with coefficients being the (graviton) cohomologies, which vanishes **identically**.
  - They are the identities of **mesons** $u_{2,3}, v_{2,3}, w_{2,3}$
  - We do not need to know how mesons are made of gluons to obtain them.

$$\sum_a g_a\, R_a = \sum_a g_a\, Q r_a = Q \sum_a g_a\, r_a = 0$$

- After constructing relations of relations, one can write them as the **Q-action on certain operators** using the (graviton) trace relations.
  - Such operators are **Q-closed** since their Q-actions vanish due to the relations of relations.
  - They can be either **Q-exact or not** and there is no trivial way to judge it easily.
  - If they are not Q-exact, they are the **black hole cohomologies**! (*)
  - Not all black hole cohomologies are constructed in this way. ($O_{n\geq 1}$ in $SU(2)$ theory)

$$O = \sum_a g_a\, r_a$$

$SU(3)$ threshold black hole cohomology

- Construct all $SU(3)$ trace relations between gravitons till $j = 20$
  - Construct all relations between these relations at $j = 24$
  → ***Q*-closed operators** at the threshold level $j = 24$

- Check their ***Q*-(non-)exactness**
  - We also found that there is only one black hole cohomology at $j = 24$ in the $SU(3)_R$ singlet sector.

R	J	#letters	#basis	#closed	#exact	#coh.	#gravitons	#BH coh.
0	4	4	1	0	0	0	0	0
$\frac{1}{2}$	$\frac{7}{2}$	5	9	1	1	0	0	0
1	3	6	91	8	8	0	0	0
$\frac{3}{2}$	$\frac{5}{2}$	7	511	85	83	2	2	0
2	2	8	1369	445	426	19	19	0
$\frac{5}{2}$	$\frac{3}{2}$	9	1898	953	924	29	28	1
3	1	10	1456	961	945	16	16	0
$\frac{7}{2}$	$\frac{1}{2}$	11	633	505	495	10	10	0
4	0	12	136	136	128	8	8	0

$SU(3)$ threshold black hole cohomology

- Unique fermionic black hole cohomology at the threshold level $j = 24$
  - scaling dimension $E = 3R + 2J = \frac{21}{2}$. (For $SU(2)$, $E = 19/2$)

$$
\begin{aligned}
O \equiv\; & -6O_6^{(0,3)} \\
=\; & 288 v^j{}_a v^{ka}{}_i \epsilon_{c_1c_2(j} \mathrm{tr}\left(\phi^{c_1}\phi^{c_2}\phi^i\psi_{k)}\right) - 72 v^a{}_b v^{bk}{}_a \epsilon_{c_1c_2(k} \mathrm{tr}\left(\phi^{c_1}\phi^{c_2}\phi^d\psi_{d)}\right) \\
& +36\epsilon_{a_1a_2i} u^{a_1k} v^{a_2}{}_j \Big[2\mathrm{tr}\left(\phi^{(i}\phi^c\phi^{j)}\psi_{(c}\psi_{k)}\right) + 2\mathrm{tr}\left(\phi^{(i|}\phi^c\phi^{|j)}\psi_{(c}\psi_{k)}\right) \\
& \qquad +9\mathrm{tr}\left(\phi^{(i}\phi^j\psi_{(c}\phi^{c)}\psi_{k)}\right) - 6\mathrm{tr}\left(\phi^{(i}\phi^{j)}\psi_{(c}\phi^c\psi_{k)}\right)\Big] \\
& -9\epsilon_{a_1a_2j} u^{a_1b} v^{a_2}{}_b \Big[2\mathrm{tr}\left(\phi^{(j}\phi^c\phi^{d)}\psi_{(c}\psi_{d)}\right) + 2\mathrm{tr}\left(\phi^{(j|}\phi^c\phi^{|d)}\psi_{(c}\psi_{d)}\right) \\
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& -20 u^{ai} v^j{}_a \epsilon_{b_1b_2b_3} \Big[2\mathrm{tr}\left(\psi_{(i}\psi_{j)}\phi^{b_1}\phi^{b_2}\phi^{b_3}\right) + \mathrm{tr}\left(\psi_{(i}\phi^{b_1}\psi_{j)}\phi^{b_2}\phi^{b_3}\right)\Big] \\
& -36 u^{ai} v^j{}_a \epsilon_{b_1b_2(i} \Big[\mathrm{tr}\left(\psi_{j)}\psi_c\phi^{b_1}\phi^{b_2}\phi^c\right) + \mathrm{tr}\left(\psi_{j)}\psi_c\phi^{b_1}\phi^c\phi^{b_2}\right) + \mathrm{tr}\left(\psi_{j)}\psi_c\phi^c\phi^{b_1}\phi^{b_2}\right)\Big] \\
& -36 u^{ai} v^j{}_a \epsilon_{b_1b_2(i} \Big[\mathrm{tr}\left(\psi_{j)}\phi^{b_1}\psi_c\phi^{b_2}\phi^c\right) + \mathrm{tr}\left(\psi_{j)}\phi^{b_1}\psi_c\phi^c\phi^{b_2}\right) + \mathrm{tr}\left(\psi_{j)}\phi^c\psi_c\phi^{b_1}\phi^{b_2}\right)\Big] \\
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& -36 u^{ai} v^j{}_a \epsilon_{b_1b_2(i} \Big[\mathrm{tr}\left(\psi_{j)}\phi^{b_1}\phi^{b_2}\phi^c\psi_c\right) + \mathrm{tr}\left(\psi_{j)}\phi^{b_1}\phi^c\phi^{b_2}\psi_c\right) + \mathrm{tr}\left(\psi_{j)}\phi^c\phi^{b_1}\phi^{b_2}\psi_c\right)\Big] \\
& +12 u^{ai} v^j{}_a \epsilon_{b_1b_2(i} \Big[5\mathrm{tr}\left(\psi_{j)}\phi^{b_1}\phi^{b_2}\right)\mathrm{tr}\left(\psi_c\phi^c\right) + 2\mathrm{tr}\left(\psi_{j)}\phi^{(b_1}\phi^{c)}\right)\mathrm{tr}\left(\psi_c\phi^{b_2}\right) - 2\mathrm{tr}\left(\psi_{j)}\phi^{b_2}\right)\mathrm{tr}\left(\psi_c\phi^{(b_1}\phi^{c)}\right)\Big]
\end{aligned}
$$

$SU(4)$ BMN black hole index

- Index over $SU(4)$ **BMN black hole cohomologies**:

$$Z - Z_{\text{grav}} = \left[-\chi_{[2,0]}(x,y)t^{28} - \chi_{[3,0]}(x,y)t^{30} + \mathcal{O}(t^{32})\right] \cdot \prod_{I<J}(1 - e^{-\Delta_I - \Delta_J})$$

- Threshold level $j = 28$
- Threshold cohomologies are not a singlet under $SU(3)_R$.

Concluding remarks

- We studied the classical cohomologies of local BPS operators in $\mathcal{N}$=4 $SU(N)$ SYM.

 - Even at small N, we can distinguish cohomologies for the gravitons and the rest.

 - The latter may describe quantum black hole microstates in AdS/CFT at finite Newton constant.

- In $SU(2)$ theory, we constructed an infinite family O_n of non-graviton cohomologies.

 - We studied partial no-hair behavior of them when graviton cohomologies are multiplied.

 - Almost all conformal primary gravitons multiplied to O_n do not appear in the index.

 - There are chances that conformal descendant gravitons may dress O_n.

 - We provide a gravity interpretation of these phenomena studying graviton hairs perturbatively superposed with non-hairy black holes ~ BPS gray galaxies.

- At higher N, we provide an ansatz for the construction of Q-closed operators utilizing the trace relations between gravitons and relations of relations.

 - They are either Q-exact or black hole cohomologies.

- For $SU(3)$, we found many towers of states (ex) BPS DDBHs) and partial no-hair behaviors and explicitly constructed the threshold black hole cohomology.

Black hole cohomologies in $\mathcal{N}$=4 SYM

$\frac{1}{16}$-BPS operators

~ Q-cohomologies

Graviton

~ monotone

Black hole

~ fortuitous

Core BH

Dressed BH

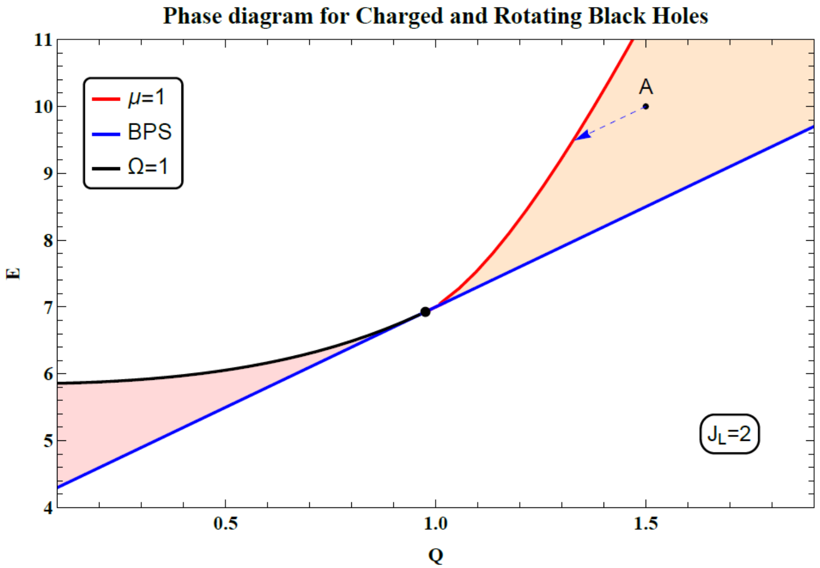
(fusion)

Large conformal descendants of gravitons

~ Grey galaxies

Large dual giants

~ DDBH



[SC, S.Kim, E.Lee, S.Lee, J.Park 23][J.Choi, SC. S.Kim, J.Lee, S.Lee 23][de Mello Koch, M.Kim, S.Kim, J.Lee, S.Lee 24]
[SC, Jain, S.Kim, Krishna, G.Kwon, E.Lee, Minwalla 25]

Future directions

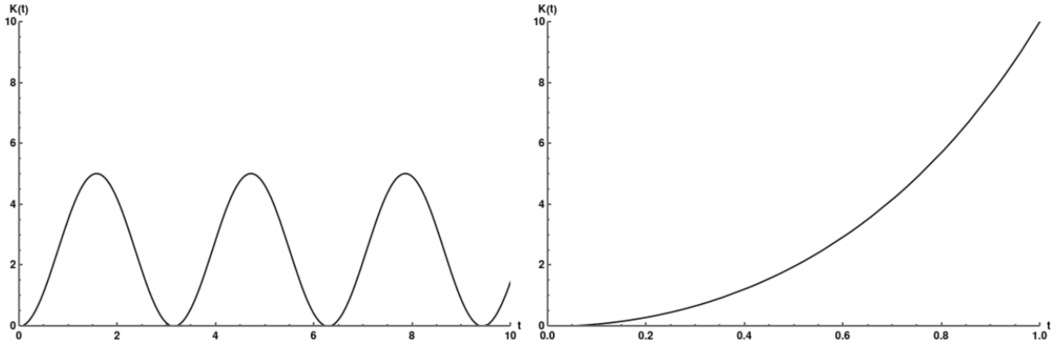
- Higher charges $j \geq 39$ for $SU(2)$?, Higher $SU(N)$?
- $\mathcal{N}$=1, 2 generalizations: implications of mass-deformation, Higgsing, S-duality, …
  - Leigh-Strassler theory [J.Choi, S.Kim 25], Klebanov-Witten theory [J.Choi, SC, S.Kim to appear], class S theories [Chang, SC, Yan WIP], …
- Other spacetime dimensions: 3d ABJ(M) theory [S.Kim, J.Lee, S.Lee, H.Oh 25] [Belin, Singh, Vadala, Zaffaroni 25], …
- BPS gray galaxies, BPS DDBHs [SC, Jain, S.Kim, Krishna, G.Kwon, E.Lee, Minwalla 25] [Mondal, V. Suryanarayana25]
  - Large N saddle points of the index
  - Grand canonical ensemble
  - Nonlinear construction: condensing BPS modes outside the horizon [SC, S.Kim, E.Lee, S.Lee, J.Park 23], connecting BH and LLM geometries? [Lin, Lunin, Maldacena 04]

Future directions

- Analyzing the supercharge cohomologies, we probed the behavior of particle excitations outside the event horizon, giving rise to the supersymmetric hairy black holes.

  - What can we learn about the interior of black holes from these supercharge cohomologies?

  - Correlation function: <HLLH> vs. <LLLL> → Operator complexity, quantum chaos, information problem, …



(a) K-Compexity $K_L(t)$ vs t for LLLL .   (b) K-Compexity $K_H(t)$ vs t for HLLH.

[Kundu, Malvimat, Sinha 23]

- Introducing the ADE singularity on AdS in this context [SC, Tachikawa 25]

  - New degrees of freedom: branes wrapping the singularity

  - Topological degrees of freedom

  - 2 kinds of phase transition at $E \sim O(N^2)$: Hawking-Page transition of AdS_5,
    deconfinement transition of tensionless strings on S^5

Thank you for listening.