Resurgence of the QCD Adler function and g-2 connection

Nagoya University January 2022

Juan Carlos Vasquez

University of Massachusetts-Amherst

.

(Work made in collaboration with Alessio Maiezza. arXiv:2104.03095 **and** 2111.06792)

AMHERST CENTER FOR FUNDAMENTAL INTERACTIONS

Physics at the interface: Energy, Intensity, and Cosmic frontiers

University of Massachusetts Amherst

Outline of the talk

1. Motivation

2. Borel and Borel-Ecalle resummation

3. Resurgence of the RGE (RRGE)

4. Bridge Equation and Resurgence

Outline of the talk

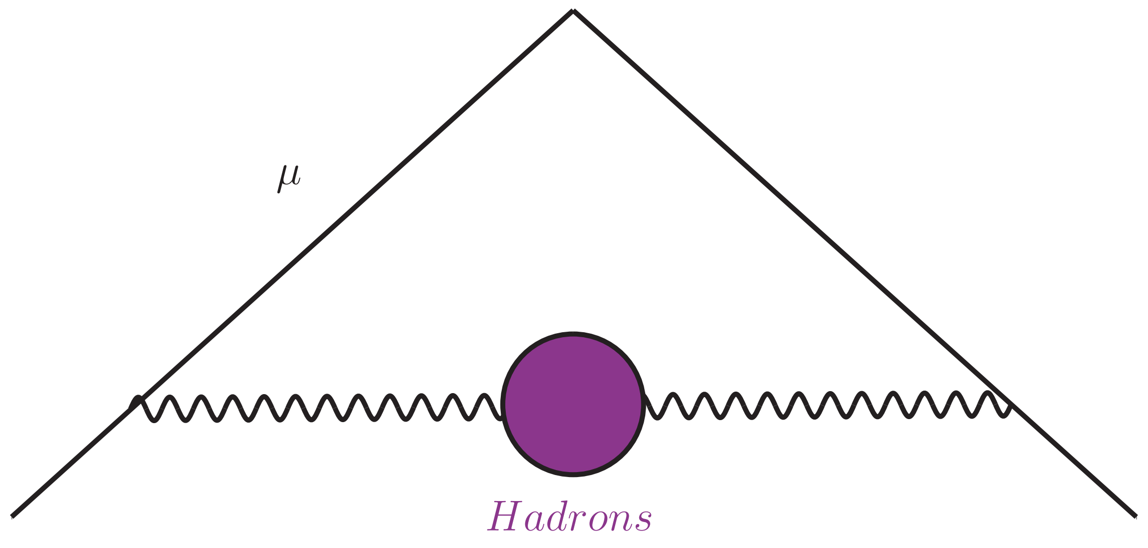
1. The Resurgence of the QCD Adler function

2. Muon g-2 connection

3. Summary and conclusions

Muon g-2 anomaly

Vacuum polarization function vs g-2

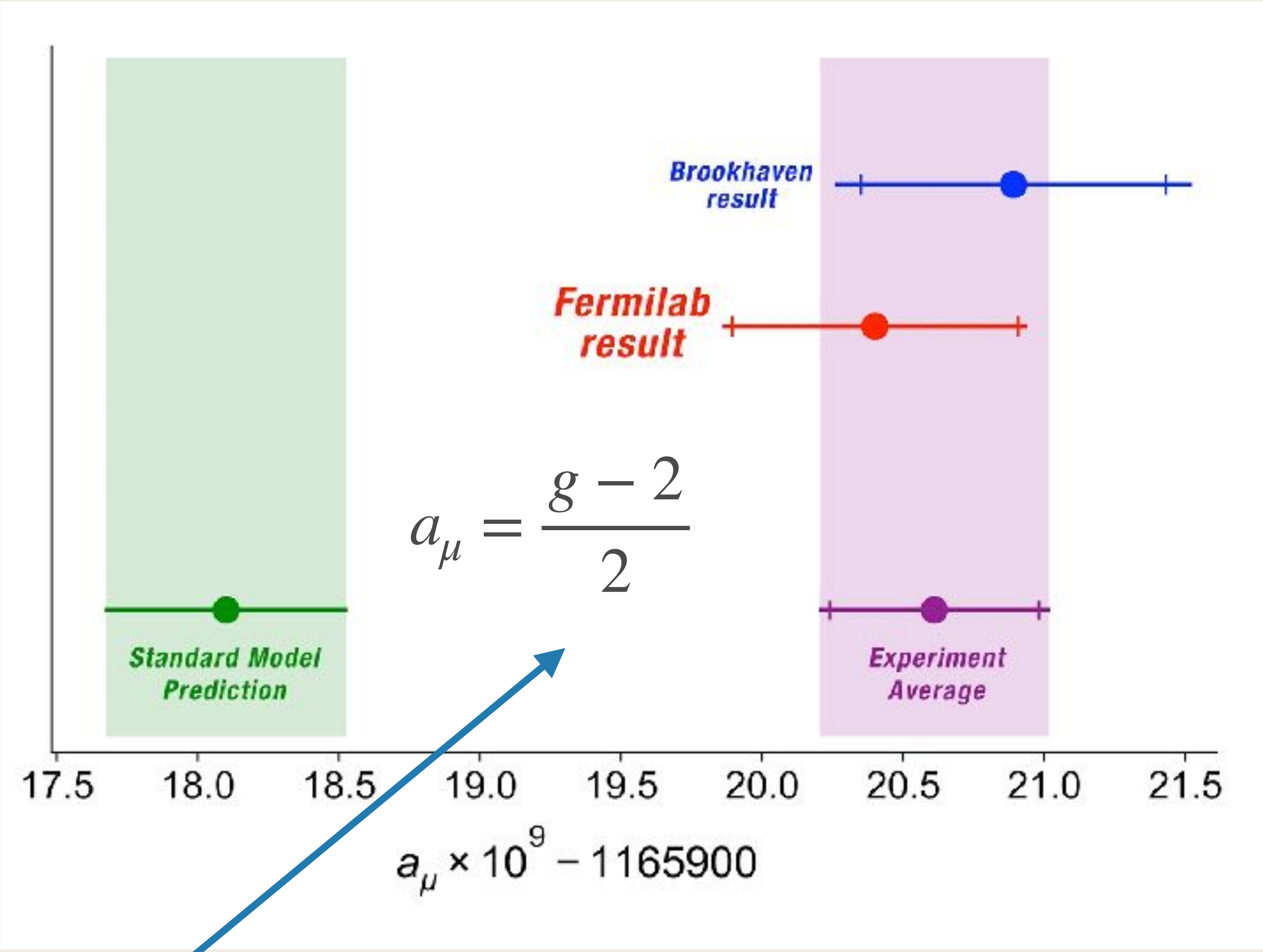


The magnetic moment of the muon $\vec{\mu}$ directed along its spin $\vec{s}$ is given by

$$\vec{\mu} = g\frac{Q_e}{2m_\mu c}\vec{s},$$

Q_e is the electric charge, m_μ is the muon mass, c is the speed of light, $g \neq 2$ at the quantum level.

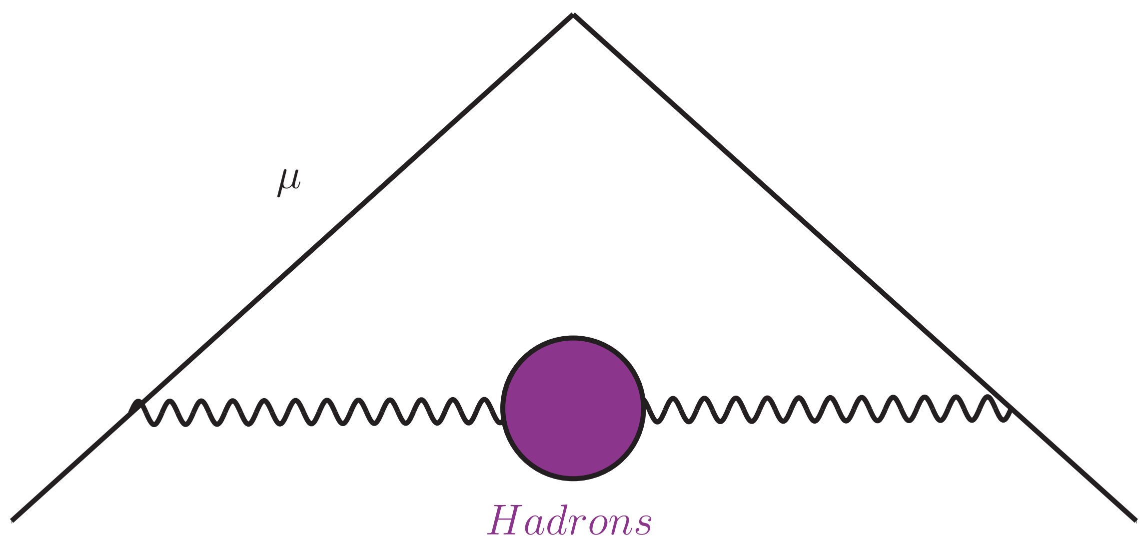
Image taken from g-2 collaboration



Can we explain the gap by new physics?

Muon g-2 anomaly

Vacuum polarization function vs g-2



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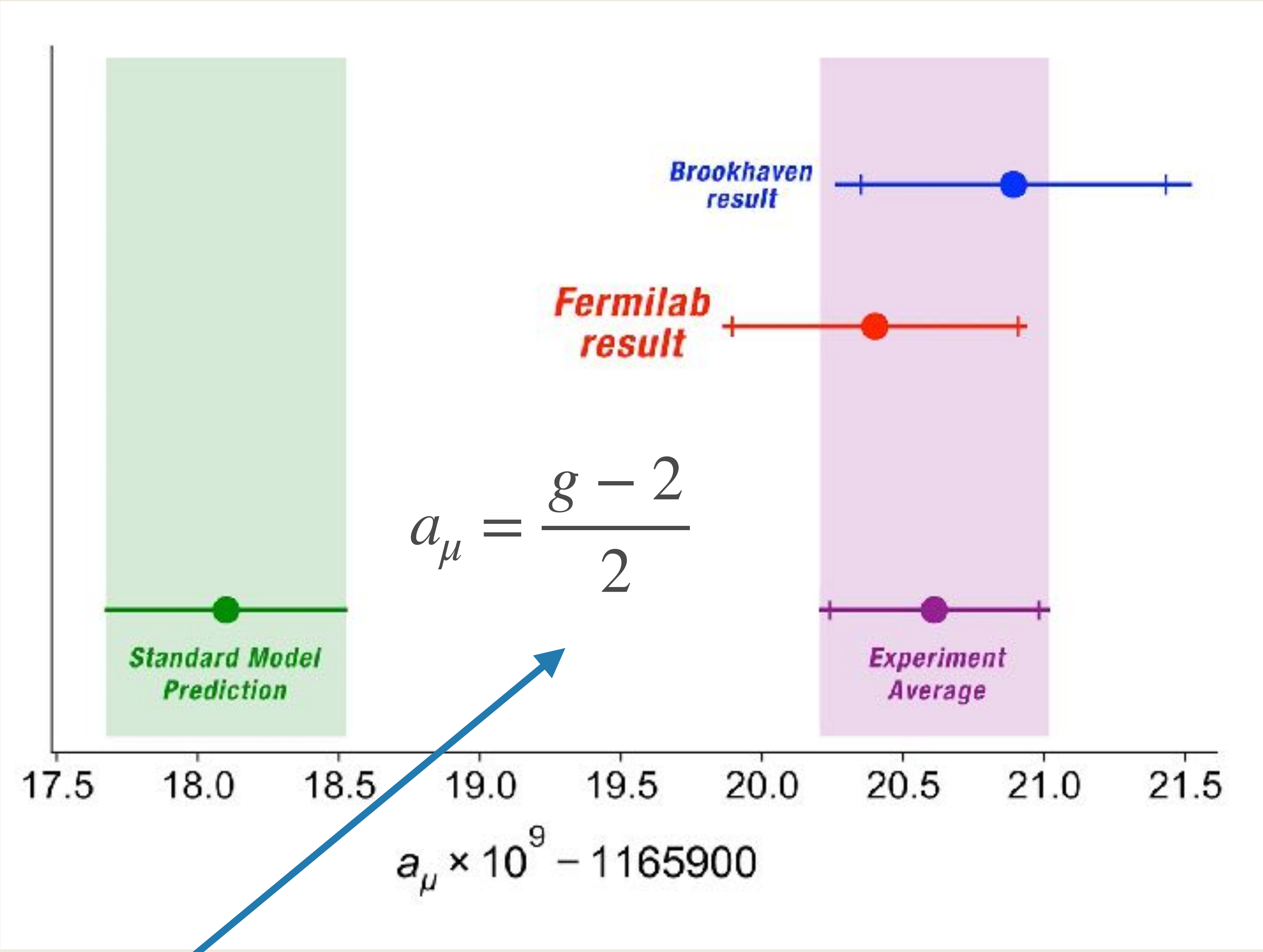

Can we explain the gap by including non-analytic Corrections in $\alpha_s(\mu)$? (Topic covered in this talk)

Motivation

Vol. 52 (2021) *Acta Physica Polonica B* No 6–7

THE BIG QUESTIONS IN
ELEMENTARY PARTICLE PHYSICS

Gerard 't Hooft

Faculty of Science, Department of Physics, Institute for Theoretical Physics
Princetonplein 5, 3584 CC Utrecht, The Netherlands
g.thooft@uu.nl
http://www.staff.science.uu.nl/~hooft101

The question *How do we sum the perturbation terms, or is there another way to obtain the exact equations for all interactions?* is correctly posed but it seems to be not so urgent. We can arrange the diagrams in such a way that diagrams calculated using perturbation theory determine with a satisfactory accuracy how the elementary particles will interact under practically all circumstances, as if we *nearly have the 'ultimate theory' at our fingertips.*

But this is not true for many reasons. First, the perturbation expansions are still formally divergent, so that we still do not quite understand what the equations are at the most fundamental level. Secondly, there is one force that can only be taken into account at the most rudimentary level: gravity. The gravitational force cannot be included in an optimal way; we return to this shortly. The third reason for concern is that there appear to be phenomena at a very large distance scale in the universe: dark matter and dark energy. These require extensions of what we know: new particles or new theories or both.

Borel Summation (or resummation)

1. Start from

$$f = \sum_{k=0}^{\infty} a_k x^{k+1}, \quad a_k \propto k!$$

Its Borel transform is ($B(x^{n+1}) = t^n/n!$)

$$\hat{f} = \sum_{k=0}^{\infty} \frac{a_k t^k}{k!},$$

If $\hat{f}$ converges, the Borel sum of f is given by

$$s_\theta(f(x)) = L \circ B(f(x)) = \int_0^{\infty e^{i\theta}} \hat{f}(t)\, e^{-t/x} dt$$

($\theta = 0$, standard Laplace)

1) If $\hat{f}$ has do not have poles in the positive real axis f is Borel sumable

$$\begin{tikzcd}
\boxed{\text{Convergent series}} \arrow[r] \arrow[d] & \boxed{\text{Summation}} \arrow[r] & \boxed{\text{Analytic functions}} \arrow[d] \\
\boxed{\text{Closure under "all" operations}} \arrow[d] & & \boxed{\text{Closure under "all" operations}} \arrow[d] \\
\boxed{\text{Transseries}} \arrow[r] & \boxed{\text{E-B Summation}} \arrow[r] & \boxed{\text{Analyzable functions}}
\end{tikzcd}$$

This is the only known way to close functions under the listed operations.

(i) Algebraic operations: addition, multiplication and their inverses.
(ii) Differentiation and integration.
(iii) Composition and functional inversion.

O. Costin, Monographs and Surveys in Pure and Applied Mathematics, Chapman and Hall/CRC, 2008.

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Borel-Ecalle summation

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($\theta = 0$, standard Laplace)

1) If $\hat{f}$ has do not have poles in the positive real axis f is Borel sumable

2) If $\hat{f}$ has do have poles in the positive real axis f is not Borel summable

This is the well-known Borel summation

$$\begin{tikzcd}
\boxed{\text{Convergent series}} \arrow[r] \arrow[d] & \boxed{\text{Summation}} \arrow[r] & \boxed{\text{Analytic functions}} \arrow[d] \\
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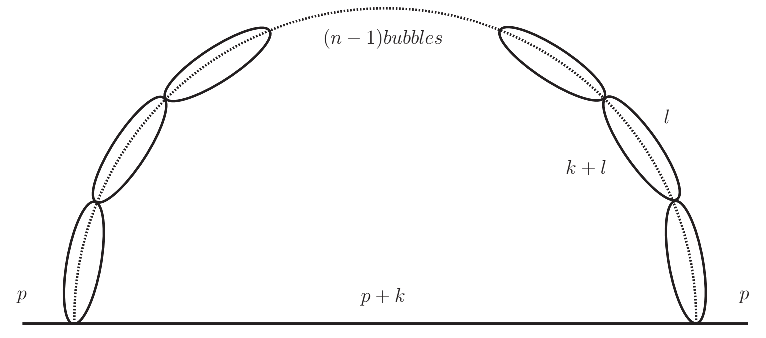
Instatons and Renormalons

Renormalons

(At least) Two problems: $n!$-behavior sources

- Instantons: these can be treated with semi-classical methods (expansions around saddle points, e.g. see[Lipatov 1977], optimal truncation,...). The semi-classicality refers to the fact that instantons are related to minimization of the classical action, and they are usually connected with tunneling (e.g. bounce solutions and vacuum decay that are indeed semi-classical calculations, see[Coleman 1977]). So they are not "dangerous objects" for QFT.
- Renormalons: deep problem, no semi-classical limit, no way to avoid the ambiguity $\Rightarrow$ they signal some inconsistency in the attempt to extend renormalization to finite values of the coupling.

As said above, because these objects (and because of the path deformation of the Laplace integral), series are turned in transseries.



$\lambda\phi(x)^4$

$\propto n!\lambda^{n+1}$

Figure: 't Hooft's skeleton diagram.

['t Hooft '79]

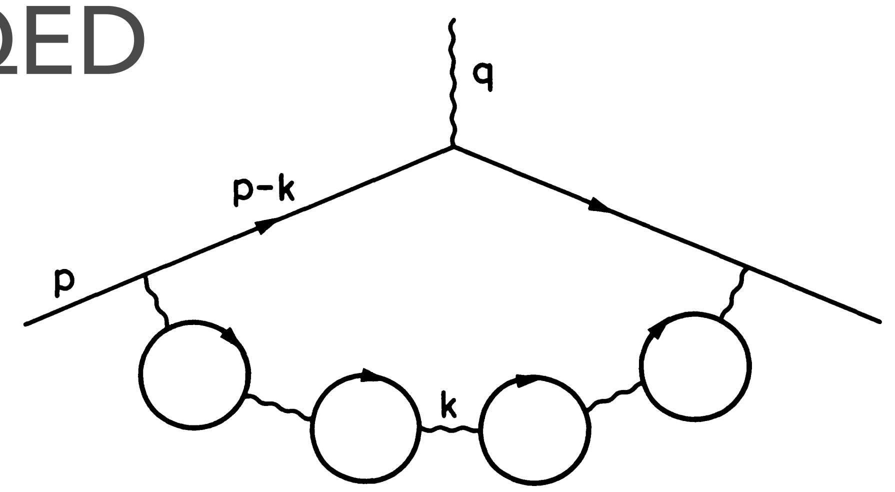


Fig. 3 Fourth member of a subclass of diagrams discussed in this section.

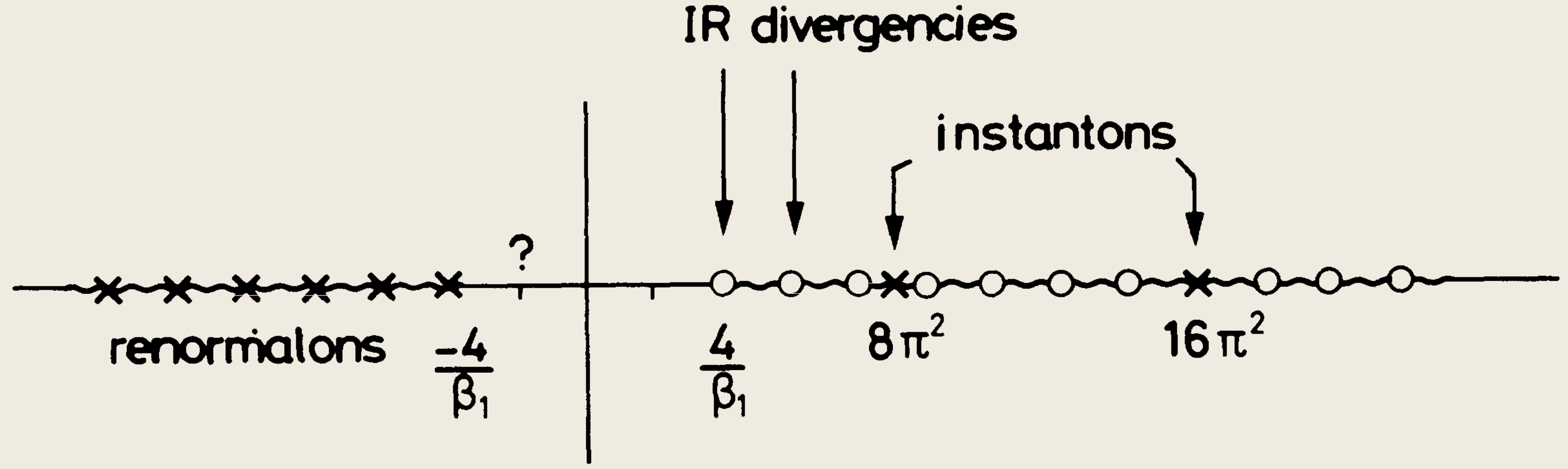


Fig. 6 Borel z plane for QCD. The circles denote IR divergences that might vanish or become unimportant in colour-free channels.

T' Hooft 1979

Key results

1. We apply the a Borel-Ecalle resummation procedure to renormalons, merging it with theory Renormalization Group.

2. Extends perturbation theory to be valid for finite coupling. PT is only valid when $\alpha_s \to 0$ (Dyson 1957)

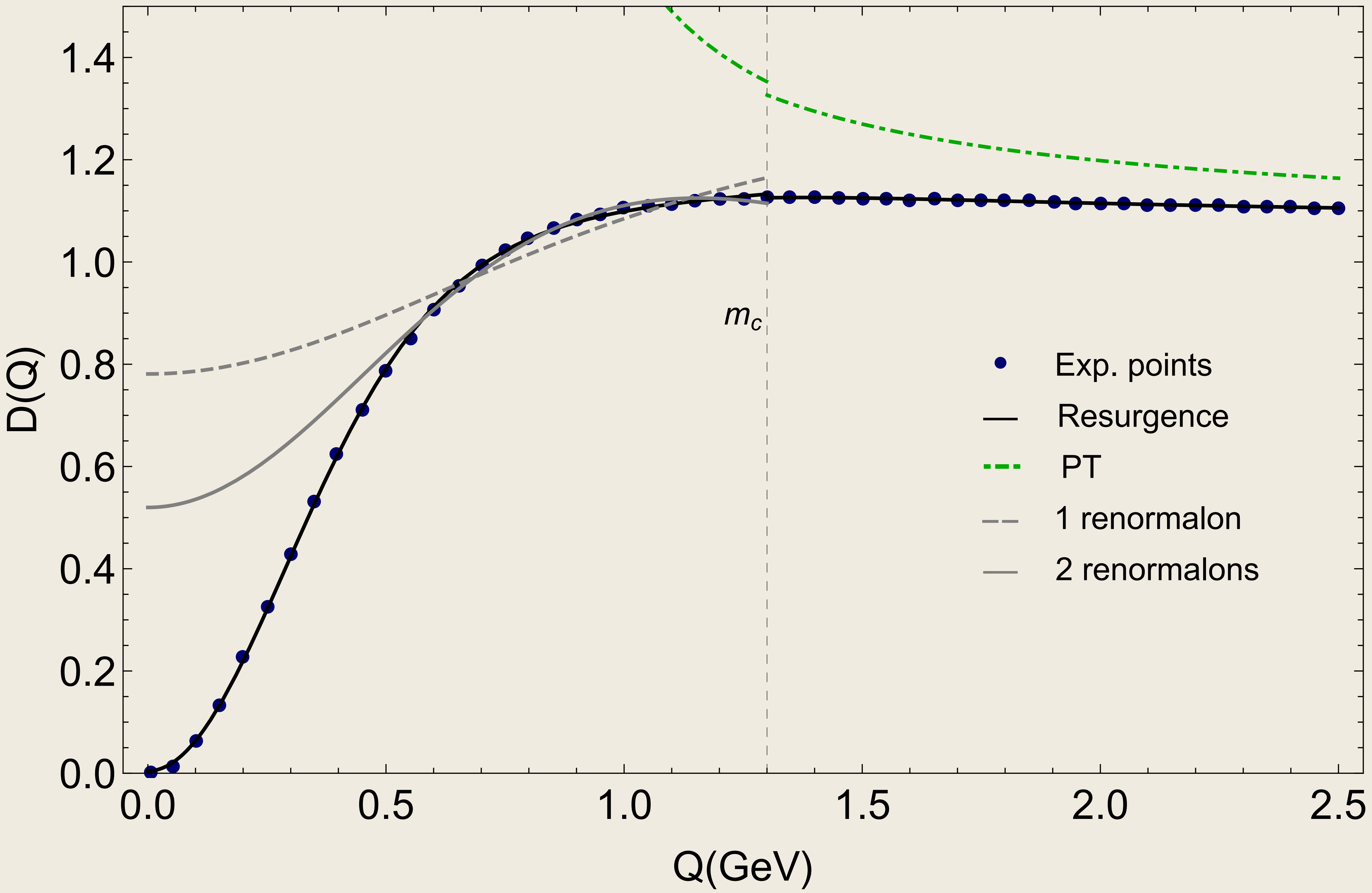
3. We get a transseries analytic expression for the QCD Adler function described by a finite number of arbitrary constants after resumming renormalons

$$D(Q^2) = D_0(Q^2) - \frac{4\pi}{\beta_0} c_1 e^{\frac{2}{\beta_0\,\alpha_s(Q^2)}} + C e^{\frac{1}{\beta_0\,\alpha_s(Q^2)}} \left(\frac{1}{\alpha_s(Q^2)} \right)^{a_p} D_1(Q^2)\,,$$

$$\boxed{\beta(\alpha_s) = \mu^2 \frac{d\alpha_s}{d\mu^2} = \beta_0 \alpha_s^2 + \beta_1 \alpha_s^3 + \mathcal{O}(\alpha_s)^4.}$$

4. We then apply these new ideas to the QCD Adler function and find we can fit the “experimental Adler function” using an effective running for the strong coupling α_s

Key result



$$\alpha_s(Q) = \frac{4\pi}{11\ln(z+\chi_g) - 2n_f \ln(z+\chi_q)/3},$$

$$z = Q^2/\Lambda^2, \quad \chi_q = 4m_q^2/\Lambda^2, \quad \chi_g = 4m_g^2/\Lambda^2,$$

Parameter	Low energy fit
K	0.80512
C	0.23957
c_1	-0.35794
Λ	697 MeV

$$-i\int d^4x e^{-iqx}\left\langle 0 \left| T\left(j_\mu(x) j_\nu(0)\right) \right| 0 \right\rangle = \left(q_\mu q_\nu - q^2 g_{\mu\nu}\right) \Pi\left(Q^2\right),$$

$$D\left(Q^2\right) = 4\pi^2 Q^2 \frac{d\Pi\left(Q^2\right)}{dQ^2},$$

$D(Q)$ extracted from $\sigma(e^+e^- \to hadrons)$

Using dispersion relations

- S. Eidelman, F. Jegerlehner , A.L. Kataev , O. Veretin (1998)
  - Published in: *Phys.Lett.B* 454 (1999) 369-380 • e-Print: hep-ph/9812521 [hep-ph]

Resurgence of the RGE

- Consider

$$\Gamma_R^{(2)} \equiv i\left(p^2 - m^2\right) G(L, \alpha_s) \qquad L = log(\mu)$$

where

$$G(L, \alpha_s) = \gamma_0(\alpha_s) + \sum_{i=1}^{\infty} \gamma_i(\alpha_s) L^i + R(\alpha_s)$$, where $R(\alpha_s) \propto n!$ (all $n!$ contributions inside $R(g)$)

- G satisfies the RGEs

$$\beta(\alpha_s) = \mu^2 \frac{d\alpha_s}{d\mu^2} = \beta_0 \alpha_s^2 + \beta_1 \alpha_s^3 + \mathcal{O}(\alpha_s)^4$$

$$\left[-\partial_L + \beta(\alpha_s)\partial_{\alpha_s} - \gamma\right] G(L, \alpha_s) = 0\,, \ \beta(\alpha_s) = \frac{d\alpha_s(\mu)}{d\log(\mu)}, \qquad \gamma(\alpha_s) = \frac{1}{2}\frac{d\log Z}{d\log(\mu)} =^{RGE} \frac{1}{2}\frac{d\log G}{d\log(\mu)}\,,$$

As it is well known one can use this equation to find the Green function at all orders in PT

RGE, Renormalons and Resurgence

- Plugging this non-perturbative $G(L, \alpha_s) = \sum_{i=0}^{\infty} \gamma_i(\alpha_s) L^i + R(\alpha_s)$ into the RGE, one get at $\mathcal{O}(L^0)$

$$R'(\alpha_s) = \frac{2(\gamma(\alpha_s) - \gamma_1(\alpha_s))}{\beta(\alpha_s)} + \frac{2\,\gamma(\alpha_s)}{\beta(\alpha_s)} R\,,$$

- Recall that in perturbation theory the 2-point function may be written as

$$G \sim \gamma_0 + \sum_{i}^{\infty} \gamma_i(\alpha_s) L^i\ ,$$

$L = \ln(-q^2/\mu^2)$ and using the renormalization condition G = 1 when $L = 0,\ \gamma_0 = 1$

RGE, Renormalons and Resurgence

- Using the results of Refereces

  - A. Maiezza and J. C. Vasquez, *Non-local Lagrangians from Renormalons and Analyzable Functions*, *Annals Phys*. 407 (2019) 78–91, [1902.05847].

  - J. Bersini, A. Maiezza and J. C. Vasquez, *Resurgence of the Renormalization Group Equation*, *Annals Phys*. 415 (2020) 168126, [1910.14507].

$$\frac{dR(\alpha_s)}{d\alpha_s} = \frac{q}{\beta_0\alpha_s^2}R(\alpha_s) + \frac{\beta_0(a_0q + a + s) - \beta_1\, q}{\beta_0^2}\frac{R(\alpha_s)}{\alpha_s} + a_0\left(\frac{a}{\beta_0} - 1\right) + \mathcal{O}(R(\alpha_s)^2)$$

$$\gamma(\alpha_s) = \gamma_1(\alpha_s) + q\, R(\alpha_s) + \frac{1}{2}(2s\alpha_s R(\alpha_s)) + \mathcal{O}(R^2\,|\,\alpha_s R),$$

$$\gamma_1(\alpha_s) = a\alpha_s + \mathcal{O}(\alpha_s^2)$$

$$\gamma_0(\alpha_s) := 1 + a_0\alpha_s + \mathcal{O}(\alpha_s^2)$$

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ODE in α_s

$$\gamma(\alpha_s) = \gamma_1(\alpha_s) + q\,R(\alpha_s) + \frac{1}{2}(2s\alpha_s R(\alpha_s)) + \mathcal{O}(R^2\,|\,\alpha_s R),$$

Non-linear in $R(\alpha_s)$

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$$\gamma_0(\alpha_s) := 1 + a_0\alpha_s + \mathcal{O}(\alpha_s^2)$$

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Position of singularities in the Borel Transform

$$\frac{dR(\alpha_s)}{d\alpha_s} = \frac{q}{\beta_0\alpha_s^2}R(\alpha_s) + \frac{\beta_0(a_0q + a + s) - \beta_1\, q}{\beta_0^2}\frac{R(\alpha_s)}{\alpha_s} + a_0\left(\frac{a}{\beta_0} - 1\right) + \mathcal{O}(R(\alpha_s)^2)$$

ODE in α_s

$$\gamma(\alpha_s) = \gamma_1(\alpha_s) + q\,R(\alpha_s) + \frac{1}{2}(2s\alpha_s R(\alpha_s)) + \mathcal{O}(R^2\,|\,\alpha_s R),$$

Non-linear in $R(\alpha_s)$

$$\gamma_1(\alpha_s) = a\alpha_s + \mathcal{O}(\alpha_s^2)$$

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$-\xi$

$$\frac{dR(\alpha_s)}{d\alpha_s} = \frac{q}{\beta_0 \alpha_s^2} R(\alpha_s) + \frac{\beta_0(a_0 q + a + s) - \beta_1 q}{\beta_0^2} \frac{R(\alpha_s)}{\alpha_s} + a_0 \left(\frac{a}{\beta_0} - 1 \right) + \mathcal{O}(R(\alpha_s)^2)$$

ODE in α_s

$$\gamma(\alpha_s) = \gamma_1(\alpha_s) + q\, R(\alpha_s) + \frac{1}{2}(2s\alpha_s R(\alpha_s)) + \mathcal{O}(R^2 \,|\, \alpha_s R),$$

Non-linear in $R(\alpha_s)$

$$\gamma_1(\alpha_s) = a\alpha_s + \mathcal{O}(\alpha_s^2)$$

$$\gamma_0(\alpha_s) := 1 + a_0 \alpha_s + \mathcal{O}(\alpha_s^2)$$

The solution to this equation is a

Transseries $R(\alpha_S) = \sum_{k=0}^{\infty} C^n R_n(\alpha_s)\, \alpha_s^{k\xi}\, e^{\frac{n}{\beta_0 \alpha_s}}$

O. Costin, Monographs and Surveys in Pure and Applied Mathematics, Chapman and Hall/CRC, 2008.

RGE, Renormalons and Resurgence

- The solution to the above non-linear equation is

$$R(\alpha_S) = \sum_{k=0}^{\infty} C^n R_n(\alpha_s)\, \alpha_s^{k\xi}\, e^{\frac{n}{\beta_0 \alpha_s}} \quad \text{(one parameter transseries)}$$

PT gives $R_0(\alpha_s)$

- The Borel transform of the solution is of the form

$$B(R(g)) \propto \sum_n \frac{1}{\left(z - \frac{n\,q}{\beta_0}\right)^{1+\xi}} \simeq \sum_n \frac{1}{\left(z - \frac{n\,q}{\beta_0}\right)^{2+\mathcal{O}(\beta_1)}}$$

from the bubble-diagrams expression then $q = 1$ and s is such that we get quadratic poles

- The above non-linear differential equation is precisely of the kind studied in

O. Costin, Monographs and Surveys in Pure and Applied Mathematics, Chapman and Hall/CRC, 2008.

RGE, Renormalons and Resurgence

- The solution to the above non-linear equation is

$$R(\alpha_S) = \sum_{k=0}^{\infty} C^n R_n(\alpha_s)\, \alpha_s^{k\xi}\, e^{\frac{n}{\beta_0 \alpha_s}}$$

HOW DO WE FIND THE FUNCTIONS $R_n(\alpha_s)$ FOR $n > 0$?

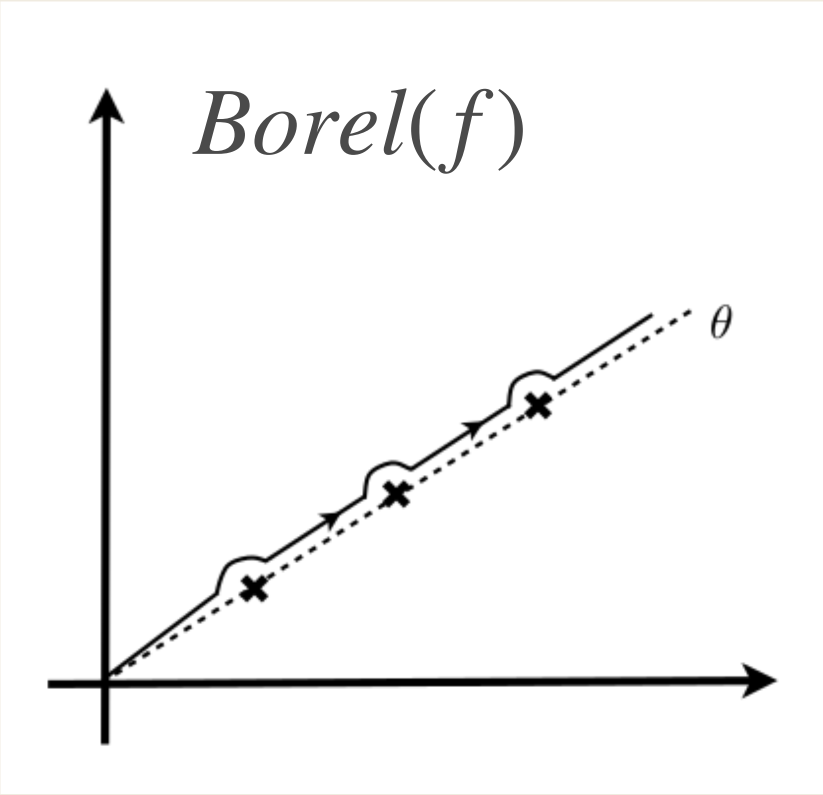
KEY CONCEPT OF “RESURGENCE”

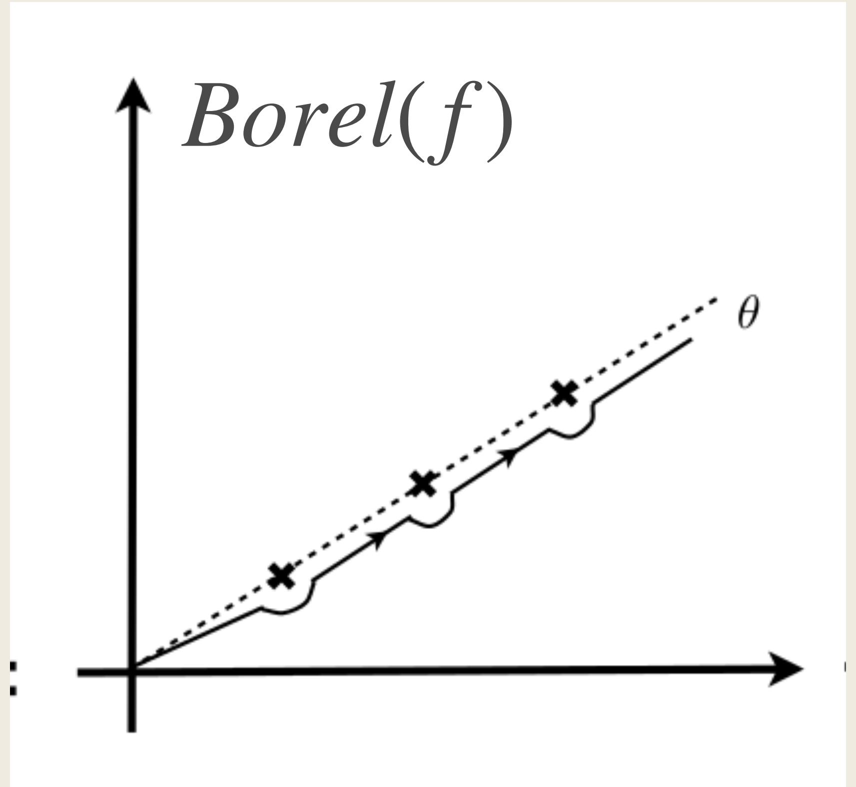
Demystifying Resurgence

1. Consider the transseries

$$f(x) = \sum_{n=0}^{\infty} f_n(x) e^{-n\lambda/x}$$

2. We are interested in the difference





$$(s_{\theta-} - s_{\theta+})f(x) = \sum_n \left(s_{\theta-} f_n - s_{\theta+} f_n\right) \cdot e^{-n\lambda/x}$$

$$s_\theta(f(x)) = L \circ B(f(x)) = \int_0^{\infty e^{i\theta}} \hat{f}(t)\, e^{-t/x} dt$$

$$s_{\theta-} = s_{\theta+} \circ \mathfrak{S}_\theta = s_{\theta+} \circ (1 + \mathrm{disc}_\theta)$$

Alien derivative

The Stokes Automorphism $\mathfrak{S}_\theta$ has the following structure

$$\mathfrak{S}_\theta = e^{\dot{\Delta}_\theta} \ , \ \dot{\Delta}_\theta \equiv \log \mathfrak{S}_\theta$$

J. Écalle, Six lectures on transseries, analysable functions and the constructive proof of Dulac's conjecture

$\dot{\Delta}_\theta$ is the Alien Derivative (it has all the properties of a derivative)

The following property holds

$\left[\dot{\Delta}_\theta, \partial_x\right] = 0 \, , \, \partial_x = \partial/\partial x$ denotes standard derivative

J. Écalle, Six lectures on transseries, analysable functions and the constructive proof of Dulac's conjecture

JUAN CARLOS VASQUEZ. EMAIL: JVASQUEZCARM@UMASS.EDU

Bridge Equation and Resurgence

Consider

$$\frac{dR(\alpha_s)}{d\alpha_s} = \frac{q}{\beta_0\alpha_s^2}R(\alpha_s) + \frac{\beta_0(a_0q+a+s)-\beta_1\,q}{\beta_0^2}\frac{R(\alpha_s)}{\alpha_s} + a_0\left(\frac{a}{\beta_0}-1\right) + \mathcal{O}(R(\alpha_s)^2)$$

Apply the Alien derivative

$$\dot{\Delta}_\theta\left(\frac{dR(\alpha_s)}{d\alpha_s}\right) = \frac{q}{\beta_0\alpha_s^2}\dot{\Delta}_\theta R(\alpha_s) + \frac{\beta_0(a_0q+a+s)-\beta_1\,q}{\beta_0^2}\frac{\dot{\Delta}_\theta R(\alpha_s)}{\alpha_s} + \dot{\Delta}_\theta\left(\cancelto{0}{a_0\left(\frac{a}{\beta_0}-1\right)}\right) + \mathcal{O}(\dot{\Delta}_\theta R(\alpha_s)^2)$$

Using $\left[\dot{\Delta}_\theta, \partial_{\alpha_s}\right] = 0$

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$$\frac{d\dot{\Delta}_\theta R(\alpha_s)}{d\alpha_s} = \frac{q}{\beta_0\alpha_s^2}\dot{\Delta}_\theta R(\alpha_s) + \frac{\beta_0(a_0q+a+s)-\beta_1\,q}{\beta_0^2}\frac{\dot{\Delta}_\theta R(\alpha_s)}{\alpha_s} + \mathcal{O}(\dot{\Delta}_\theta R(\alpha_s)^2)$$

Bridge Equation and Resurgence

Consider again

$$\frac{dR(\alpha_s)}{d\alpha_s} = \frac{q}{\beta_0\alpha_s^2}R(\alpha_s) + \frac{\beta_0(a_0q + a + s) - \beta_1\, q}{\beta_0^2}\frac{R(\alpha_s)}{\alpha_s} + a_0\left(\frac{a}{\beta_0} - 1\right) + \mathcal{O}(R(\alpha_s)^2)$$

Apply the derivative with respect to the one parameter transseries ($\partial_C \equiv \partial/\partial_C$)

$$\frac{d\partial_C R(\alpha_s)}{d\alpha_s} = \frac{q}{\beta_0\alpha_s^2}\partial_C R(\alpha_s) + \frac{\beta_0(a_0q + a + s) - \beta_1\, q}{\beta_0^2}\frac{\partial_C R(\alpha_s)}{\alpha_s} + \mathcal{O}(\partial_C R(\alpha_s)^2)$$

One-parameter transseries

$$\boxed{R(\alpha_S) = \sum_{k=0}^{\infty} C^n R_n(\alpha_s)\,\alpha_s^{k\xi}\, e^{\frac{n}{\beta_0\alpha_s}}}$$

Compare with

$$\frac{d\dot{\Delta}_\theta R(\alpha_s)}{d\alpha_s} = \frac{q}{\beta_0\alpha_s^2}\dot{\Delta}_\theta R(\alpha_s) + \frac{\beta_0(a_0q + a + s) - \beta_1\, q}{\beta_0^2}\frac{\dot{\Delta}_\theta R(\alpha_s)}{\alpha_s} + \mathcal{O}(\dot{\Delta}_\theta R(\alpha_s)^2)$$

Both $\dot{\Delta}_\theta R(\alpha_s)$ and $\partial_C R(\alpha_s)$

Satisfy the same ODE

then

$\dot{\Delta}_\theta R(\alpha_s) = A_\theta\, \partial_C R(\alpha_s)$ Ecalle Brigde Equation. A_θ Holomorphic constant

Bridge Equation and Resurgence

WE CAN FIT A_θ FROM DATA

DIFFICULT TO CALCULATE FOR INSTANTONS

SEE DORIGONI, SCIAPPA REVIEWS

AND IMPOSIBLE FOR RENORMALONS

T'HOOFT (1979), ZINN-JUSTIN

MAIEZZA, VASQUEZ

then

$\dot{\Delta}_\theta R(\alpha_s) = A_\theta \, \partial_C R(\alpha_s)$ Ecalle Brigde Equation. A_θ Holomorphic constant

Resurgence

$\dot{\Delta}_\theta R(\alpha_s) = A_\theta\, \partial_C R(\alpha_s)$ Ecalle Brigde Equation

Plugging $R(\alpha_S) = \sum_{k=0}^{\infty} C^K R_k(\alpha_s)\, \alpha_s^{k\xi}\, e^{\frac{k}{\beta_0 \alpha_s}}$ above and equaling the powers of $C^n\, \alpha_s^{n\xi}\, e^{\frac{n}{\beta_0 \alpha_s}}$ in each side

$\dot{\Delta}_\theta R_n(\alpha_s) = (n+1)\, A_\theta\, \alpha_s^{\xi}\, e^{\frac{1}{\beta_0 \alpha_s}}\, R_{n+1}(\alpha_s),$ in particular $\dot{\Delta}_\theta R_0(\alpha_s) = A_\theta\, \alpha_s^{\xi}\, e^{\frac{1}{\beta_0 \alpha_s}}\, R_1(\alpha_s)$ and so on …

This is Resurgence

Resurgence

conjugacy).

The Bridge Equation owes its name to the fact that it makes manifest an unexpected link between the *ordinary* and *alien derivatives* of a local object's formal integral(s). Its scope is stupendous; in fact it is virtually coextensive with "resonance" understood in the

104 *J. Écalle*

broadest possible sense, including in particular "trivial resonance" (i.e. $\lambda_i = 0$ or $\ell_i = 1$ or $\ell_i =$ unit root). If we now recall the translatability of even high-order differential equations, linear or not, into time-independent, first-order differential systems, which themselves are equivalent to vector fields; and if we further bear in mind that non-trivial Newton polygons (in differential equations) induce vanishing multipliers (in the vector field), we may grasp why the *overwhelming majority of singular differential equations also fall within the jurisdiction of resurgence, alien calculus, and the Bridge Equation.*

Generalized Borel-Laplace resummation: Resurgence

O. Costin, Monographs and Surveys in Pure and Applied Mathematics, Chapman and Hall/CRC, 2008.

- 3. Resurgence: once $Y_0(z)$ is known, the functions $Y_k(z)$ are given by

$$S_0^k Y_k = \left(Y_0^- - Y_0^{-(k-1)+}\right) \circ \tau_k,\ \tau_k : z \to z + k$$

where

$$Y_k^{-m+} = Y_k^+ + \sum_{j=1}^{m} \binom{k+j}{k} S_0^j Y_{k+j}^+ \circ \tau_{-j}\,.$$

4. The balanced average

$$Y_k^{bal} \equiv Y_k^+ + \sum_{n=1}^{\infty} 2^{-n} \left(Y_k^- - Y_k^{-n-1+}\right).$$

(Borel$(R_n(\alpha_s)) = Y_n$ and $1/\alpha_s = x$ in Costin's book)

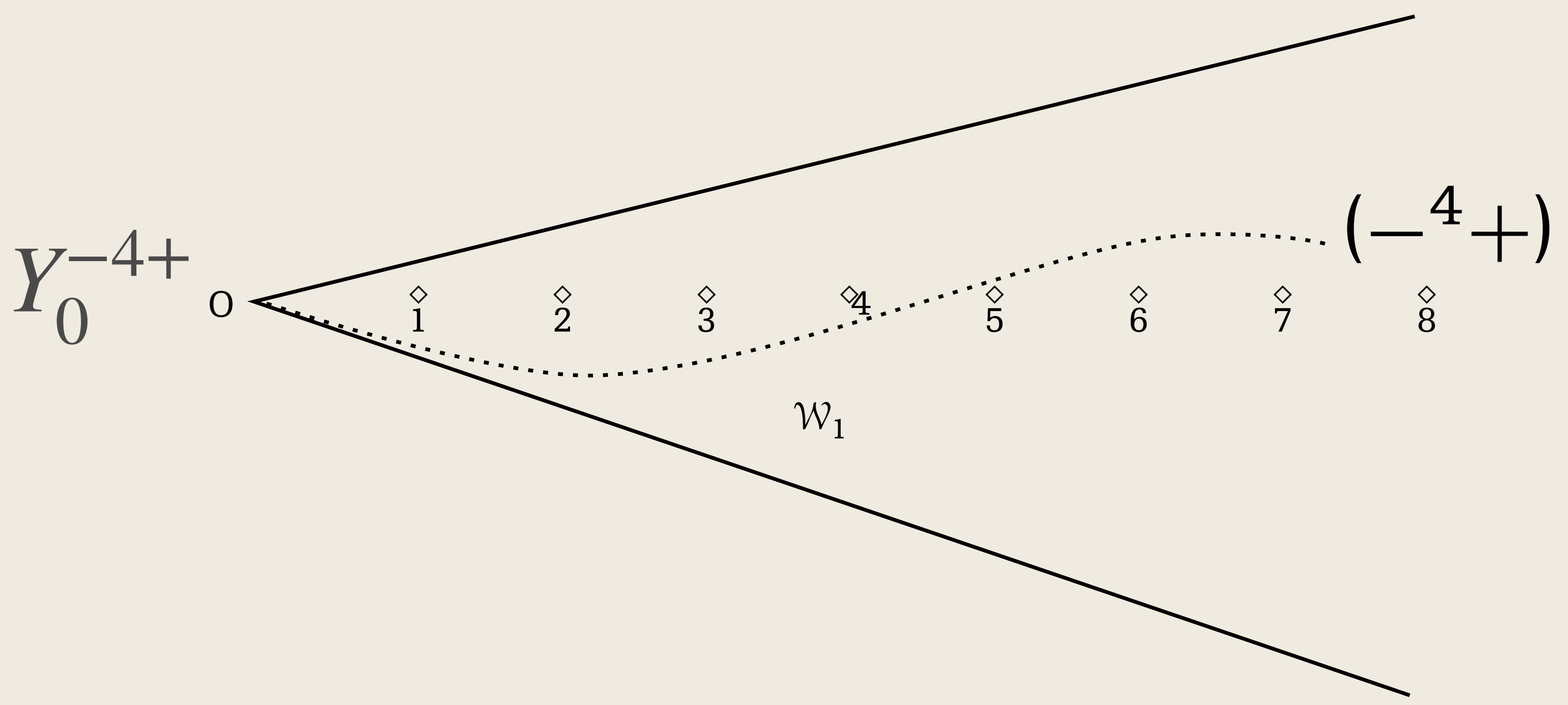


Image taken from Costin 1995

This definition preserves reality in the sense that when $y_0(g)$ is a formal series with real coefficients, then the functions y_k^{bal} are also real $\forall k$. (Costin 2008)

This operation unlike analytic continuation commutes with convolutions.

Generalized Borel-Laplace resummation: Resurgence

- The Laplace transform: when $B(R_n)$ has poles in the positive real axis, the Laplace transform is modified as follows

$$\mathscr{E}\left(R_k\right) = \mathscr{L} \circ \mathscr{B}\left(R_k\right) = \mathscr{L}\left(R_k\right) = \int_0^{\infty} B(R_k)^{bal} e^{-z/\alpha_s} dz,$$

where the balanced average guaranteed that the reality condition is satisfied

- In the mathematical literature $1/\alpha_s \to x$, so the asymptotic expansions when $x \to \infty$ correspond to the weak coupling limit $\alpha_s \to 0$.

O. Costin, Monographs and Surveys in Pure and Applied Mathematics, Chapman and Hall/CRC, 2008.

The Adler function

Consider the correlation function of two massless quark currents $j_\mu = \bar{q}\gamma_\mu q$

$$-\mathrm{i}\int d^4 x e^{-iqx}\left\langle 0\left|T\left(j_\mu(x)j_\nu(0)\right)\right|0\right\rangle = \left(q_\mu q_\nu - q^2 g_{\mu\nu}\right)\Pi\left(Q^2\right)\ ,$$

Where $Q^2 = -q^2$

The Adler function is defined as

$$D\left(Q^2\right) = 4\pi^2 Q^2 \frac{\mathrm{d}\Pi\left(Q^2\right)}{\mathrm{d}Q^2}\ ,$$

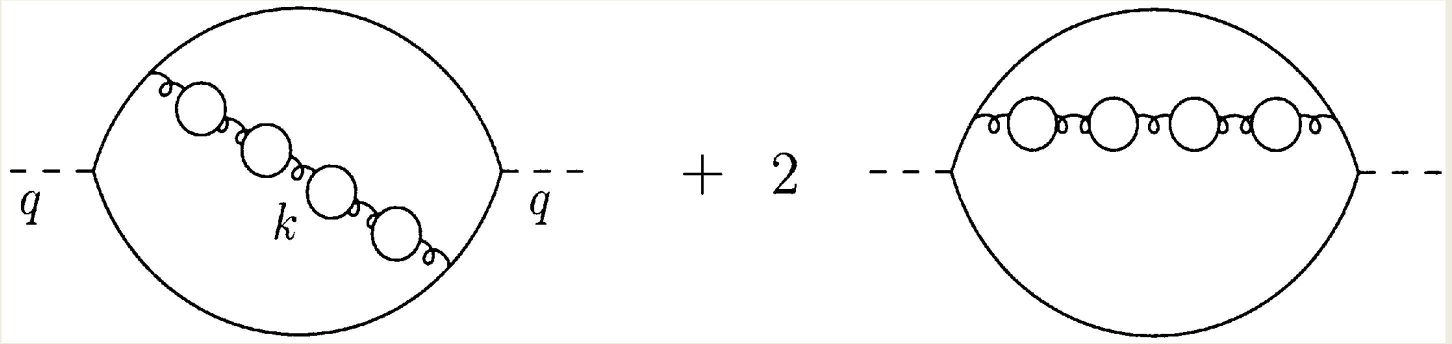
This function enters in the $R_{e^+e^-}$ ratio, hadronic τ decays and in the

Hadronic vacuum polarization contributions of the $g-2$ anomaly

The Adler function and Resurgence

The Adler function is given by

$$D\left(Q^2\right) = 4\pi^2 Q^2 \frac{\mathrm{d}\Pi\left(Q^2\right)}{\mathrm{d}Q^2},$$



Renormalon diagrams

And it can be written in perturbation theory as

$$D_{pert}\left(Q^2\right) = 1 + \frac{\alpha_s}{\pi}\sum_{n=0}^{\infty}\alpha_s^n\left[d_n\left(-\beta_0\right)^n + \delta_n\right].$$ (Divergent)

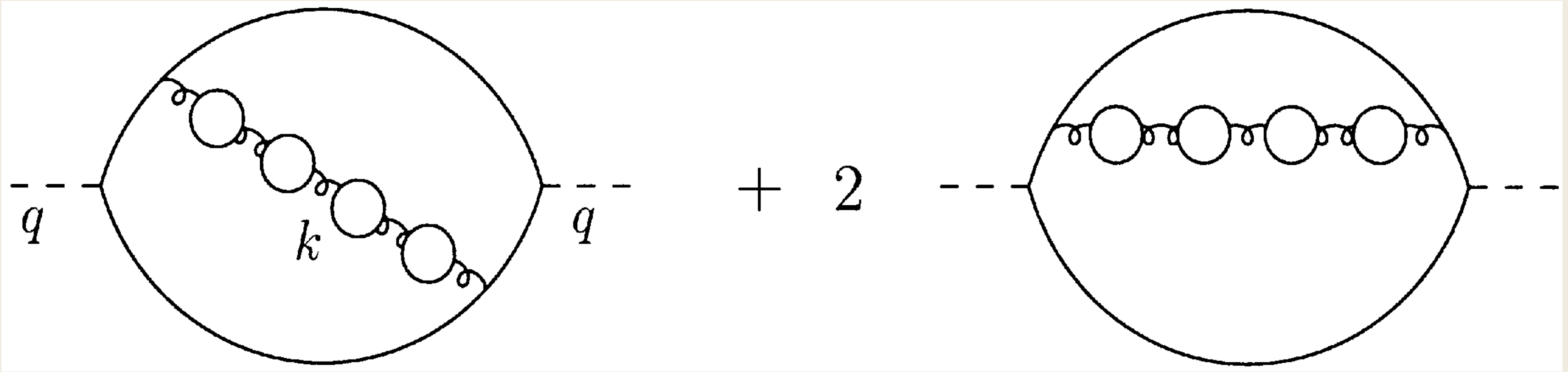
Where $d_n \propto n!$ and $\beta(\alpha_s) = \mu^2 \frac{d\alpha_s}{d\mu^2} = \beta_0\alpha_s^2 + \beta_1\alpha_s^3 + \mathcal{O}(\alpha_s)^4$

The perturbative expression is known up to $n = 3$

- S. G. Gorishnii, A. L. Kataev and S. A. Larin, *The* $O(\alpha_s^3)$*-corrections to* $\sigma_{tot}(e^+e^- \to$ hadrons) *and* $\Gamma(\tau^- \to \nu_\tau +$ hadrons) *in QCD*, *Phys. Lett. B* 259 (1991) 144–150.
- L. R. Surguladze and M. A. Samuel, *Total hadronic cross-section in e+ e- annihilation at the four loop level of perturbative QCD*, *Phys. Rev. Lett.* 66 (1991) 560–563.
- A. L. Kataev and V. V. Starshenko, *Estimates of the higher order QCD corrections to R(s), R(tau) and deep inelastic scattering sum rules*, *Mod. Phys. Lett. A* 10 (1995) 235–250, [hep-ph/9502348].

Naive non-abelianization

$$D_{pert}\left(Q^2\right) = 1 + \frac{\alpha_s}{\pi}\sum_{n=0}^{\infty}\alpha_s^n\left[d_n\left(-\beta_0\right)^n + \delta_n\right].$$



1. Naive Non-abelianization is a model for the high order behavior (Beneke. *Phys.Rept.* 317 (1999) 1-142 • e-Print: hep-ph/9807443 [hep-ph])
2. In practice it means:

I) We use the known perturbation theory expression of the Adler function up to $\mathcal{O}(\alpha_s^4)$
II) For Higher loop correction one assumes the fermion bubble-diagrams dominate i.e.

$\delta_n \sim 0$ for $n \geq 4$ and d_n is given by evaluating the bubble diagrams so that

$$d_n \propto K n!$$

Where K is an arbitrary constant
(Beneke. *Phys.Rept.* 317 (1999) 1-142 • e-Print: hep-ph/9807443 [hep-ph])

The Adler function and Resurgence

1. Using the Borel-Ecalle resummation procedure explained

O. Costin, Monographs and Surveys in Pure and Applied Mathematics, Chapman and Hall/CRC, 2008.

we get

$$D(Q^2) = D_0(Q^2) - \frac{4\pi}{\beta_0} c_1 e^{\frac{2}{\beta_0\,\alpha_s(Q^2)}} + C e^{\frac{1}{\beta_0\,\alpha_s(Q^2)}} \left(\frac{1}{\alpha_s(Q^2)}\right)^{a_p} D_1(Q^2)\,,$$

Perturbative + $K\,n!$ Contributions using Borel transform plus Cauchy principal value prescription. The constant K is fitted to data

The Adler function and Resurgence

1. Using the Borel-Ecalle resummation procedure of

O. Costin, Monographs and Surveys in Pure and Applied Mathematics, Chapman and Hall/CRC, 2008.

we get

$$D(Q^2) = D_0(Q^2) - \frac{4\pi}{\beta_0} c_1 e^{\frac{2}{\beta_0\, \alpha_s(Q^2)}} + C e^{\frac{1}{\beta_0\, \alpha_s(Q^2)}} \left(\frac{1}{\alpha_s(Q^2)}\right)^{a_p} D_1(Q^2)\,,$$

Non-perturbative ambiguity due to the first simple-pole Renormalons

Constant c_1 is arbitrary. We fix c_1 the best fit to “experimental Adler function”

The Adler function and Resurgence

1. Using the Borel-Ecalle resummation procedure of

O. Costin, Monographs and Surveys in Pure and Applied Mathematics, Chapman and Hall/CRC, 2008.

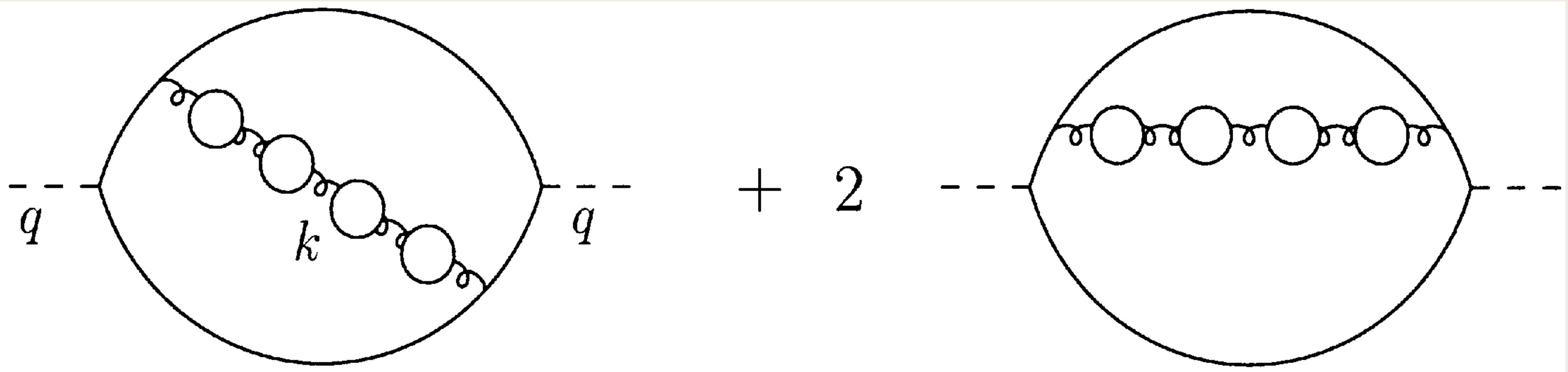
we get

$$D(Q^2) = D_0(Q^2) - \frac{4\pi}{\beta_0} c_1 e^{\frac{2}{\beta_0 \alpha_s(Q^2)}} + C e^{\frac{1}{\beta_0 \alpha_s(Q^2)}} \left(\frac{1}{\alpha_s(Q^2)} \right)^{a_p} D_1(Q^2)\,,$$

Resurgent contribution from quadratic poles. One arbitrary constant C fitted to data and one arbitrary constant K in $D_1(Q^2)$ due to resurgence relations

The Adler function and Resurgence

1. Resumming these diagrams



$$D(Q^2) = D_0(Q^2) - \frac{4\pi}{\beta_0} c_1 e^{\frac{2}{\beta_0 \alpha_s(Q^2)}} + C e^{\frac{1}{\beta_0 \alpha_s(Q^2)}} \left(\frac{1}{\alpha_s(Q^2)} \right)^{a_p} D_1(Q^2) \, ,$$

$$D_1\left(Q^2\right) = \frac{8\pi K}{3\alpha_s \beta_0^2} \left[2e^{\frac{1}{\alpha_s\beta_0}} - \left(e^{\frac{1}{\alpha_s\beta_0}} + 1 \right) \log\left(1 - e^{\frac{2}{\alpha_s\beta_0}} \right) - 2\left(e^{\frac{1}{\alpha_s\beta_0}} + 1 \right) \tanh^{-1}\left(e^{\frac{1}{\alpha_s\beta_0}} \right) \right]$$

The Adler function and Resurgence

1. Using the Borel-Ecalle resummation procedure of

O. Costin, Monographs and Surveys in Pure and Applied Mathematics, Chapman and Hall/CRC, 2008.

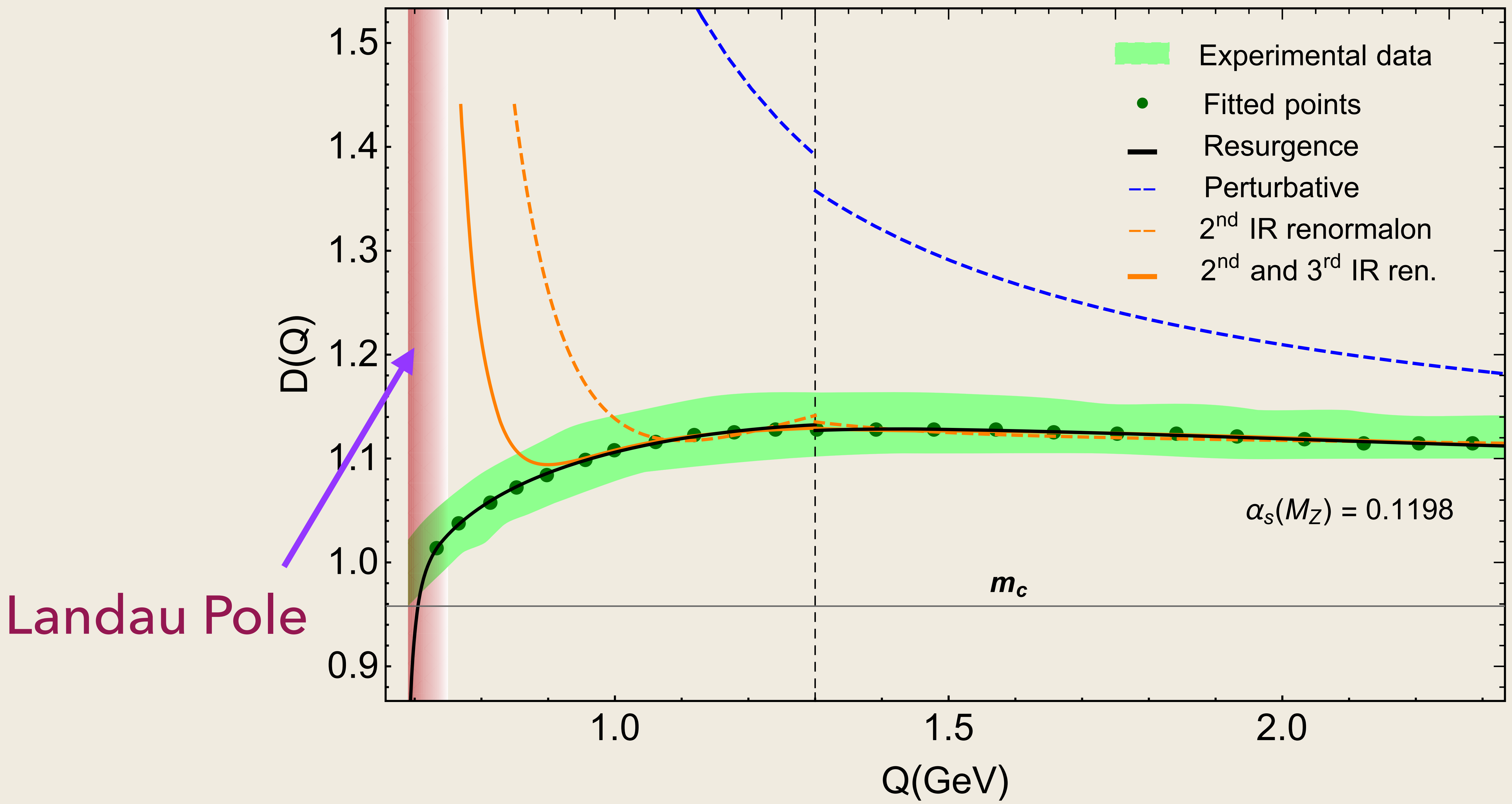
we get

$$D(Q^2) = D_0(Q^2) - \frac{4\pi}{\beta_0} c_1 e^{\frac{2}{\beta_0 \alpha_s(Q^2)}} + C e^{\frac{1}{\beta_0 \alpha_s(Q^2)}} \left(\frac{1}{\alpha_s(Q^2)} \right)^{a_p} D_1(Q^2) \, ,$$

In summary we fit to data three constants K, c_1 and C

Key result (with Landau pole)

- *Phys.Lett.B* 817 (2021) 136338 • e-Print: 2104.03095 [hep-ph]



[AM-Vasquez '21]

$$C, K, c_1 \simeq \begin{cases} -0.023, 1.41, -0.51 & Q < m_c, \\ -8.88, 0.99, -5.27 & Q \geq m_c. \end{cases}$$

We find good fit to data up to $E \sim 0.7$ GeV where the Landau Pole breaks the description

Around this scale, the coupling diverges and the transseries expansion ceases to work.

The problem of the Landau pole

The problem of the IR Landau pole

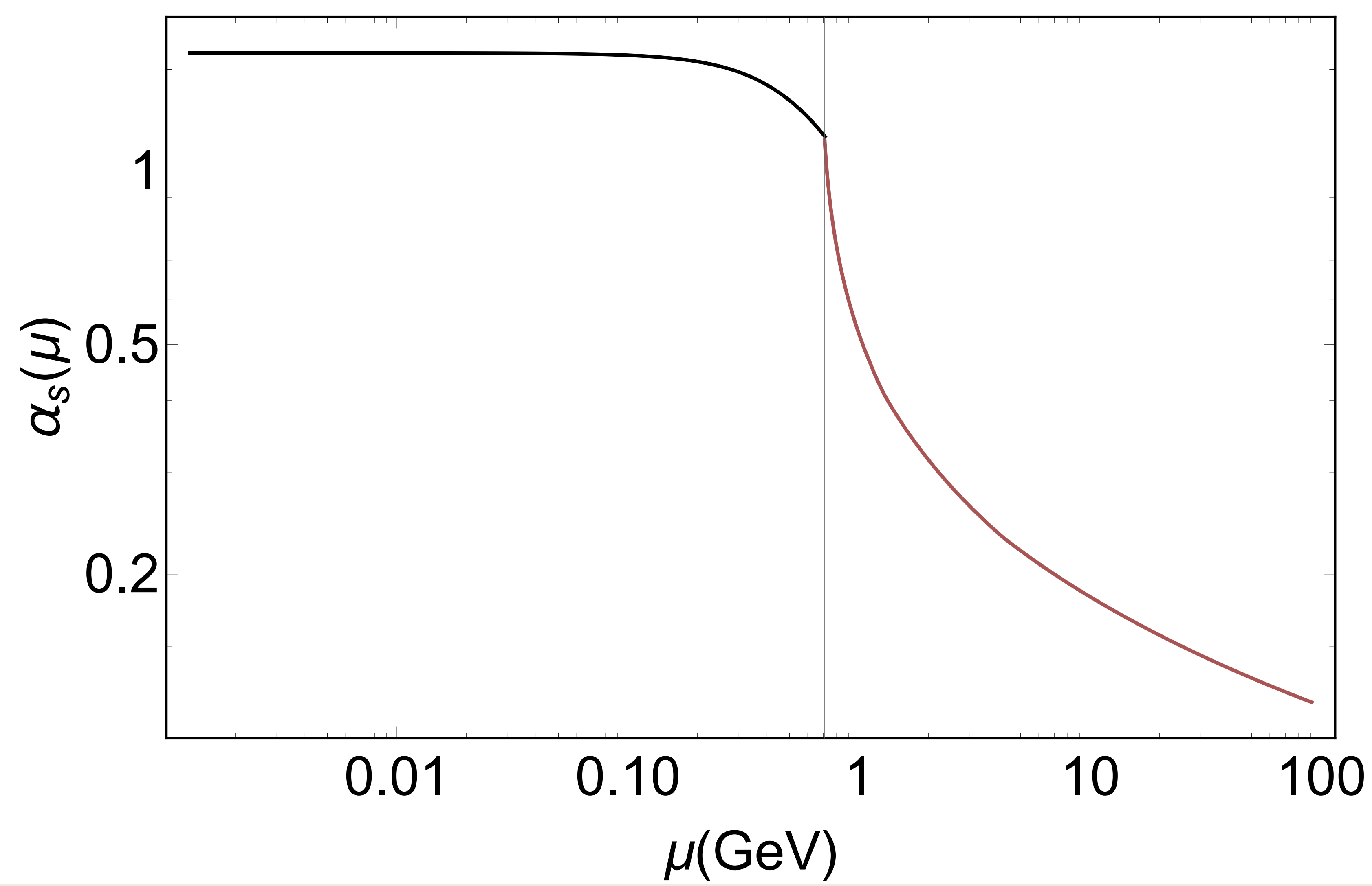
We saw that the theoretical expression follows the experimental one up to the IR Landau pole - there, things stop working because the coupling explode, but not because there is some of wrong in the resurgent procedure *per se*.

Effective solution $\Rightarrow$

Effective running for α_s.
The simplest realization is to employ Cornwall's coupling:

$$\alpha_s(Q) = \frac{4\pi}{11 \ln(z+\chi_g) - 2n_f \ln(z+\chi_q)/3},$$

[Cornwall '81, Papavassiliou-Cornwall '91]



The problem of the Landau pole

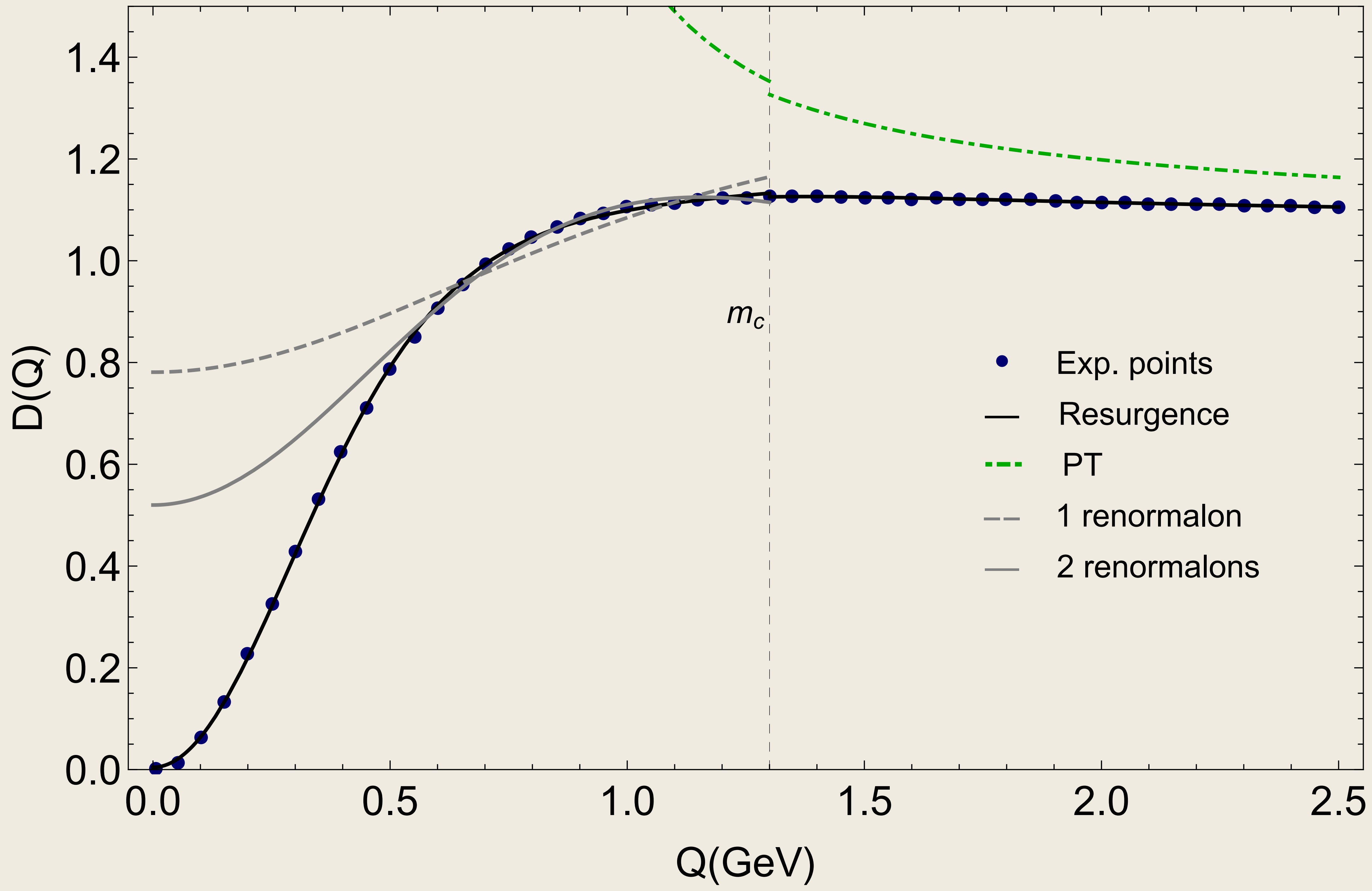
The problem of the IR Landau pole

$$\alpha_s(Q) = \frac{4\pi}{11\ln\left(z+\chi_g\right) - 2n_{\mathrm{f}}\ln\left(z+\chi_q\right)/3},$$

where $z = Q^2/\Lambda^2$, n_f is the number of flavors, $\chi_g = 4m_g^2/\Lambda^2$, $\chi_q = 4m_q^2/\Lambda^2$, the light constituent quark mass $m_q = 350$ MeV, the gluon mass $m_g \simeq 500$ MeV, and Λ denotes the QCD hadronic (non-perturbative) scale.

Possibility to describe also the running within our approach?

Key result



$$D(Q^2) = D_0(Q^2) - \frac{4\pi}{\beta_0} c_1 e^{\frac{2}{\beta_0 \alpha_s(Q^2)}} + C e^{\frac{1}{\beta_0 \alpha_s(Q^2)}} \left(\frac{1}{\alpha_s(Q^2)} \right)^{a_p} D_1(Q^2),$$

$$\alpha_s(Q) = \frac{4\pi}{11 \ln(z + \chi_g) - 2n_f \ln(z + \chi_q)/3},$$

$$z = Q^2/\Lambda^2, \quad \chi_q = 4m_q^2/\Lambda^2, \quad \chi_g = 4m_g^2/\Lambda^2$$

Parameter	Low energy fit
K	0.80512
C	0.23957
c_1	-0.35794
Λ	697 MeV

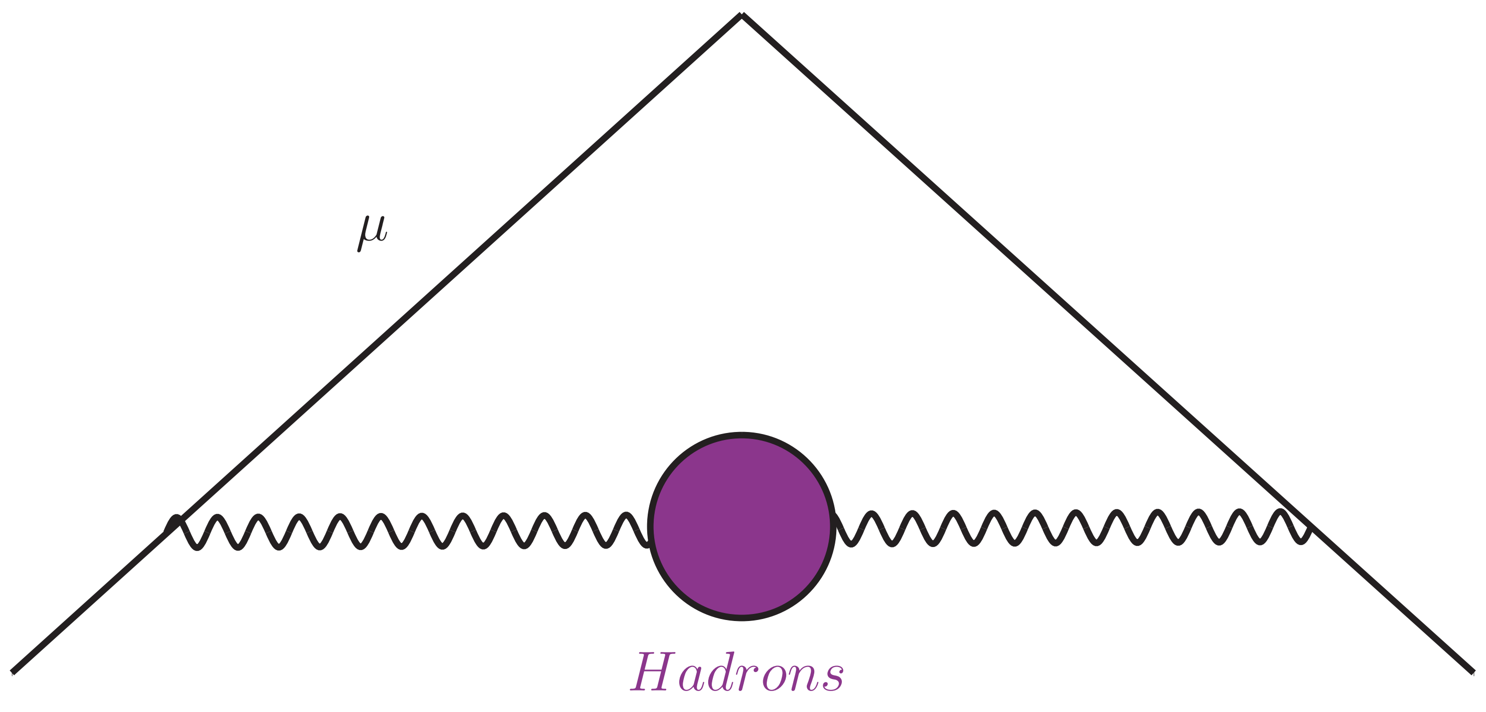
$D(Q)$ extracted from $\sigma(e^+e^- \to hadrons)$

Using dispersion relations

- S. Eidelman, F. Jegerlehner , A.L. Kataev , O. Veretin (1998)
  - Published in: *Phys.Lett.B* 454 (1999) 369-380 • e-Print: hep-ph/9812521 [hep-ph]

Muon g-2

Vacuum polarization function vs g-2



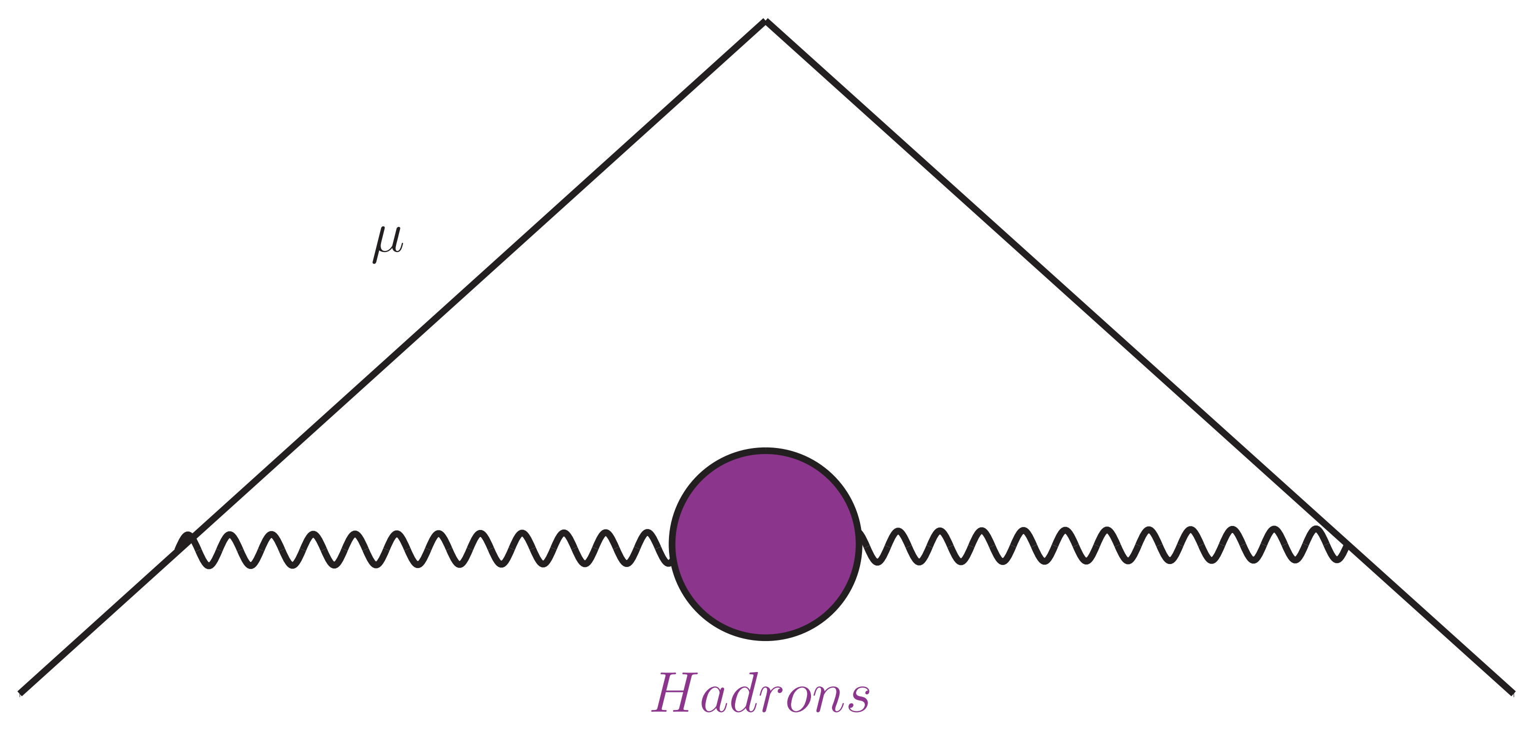
The magnetic moment of the muon $\vec{\mu}$ directed along its spin $\vec{s}$ is given by

$$\vec{\mu} = g\frac{Q_e}{2m_\mu c}\vec{s},$$

Q_e is the electric charge, m_μ is the muon mass, c is the speed of light, $g \neq 2$ at the quantum level.

Muon g-2

Vacuum polarization function vs g-2



$$a_\mu = (g-2)/2$$

$$a_\mu^{(\text{h.v.p.})} = 2\pi^2 \left(\frac{\alpha}{\pi}\right)^2 \int_0^1 \frac{dx}{x}(1-x)(2-x) D\left(Q\right),$$

$$Q = \sqrt{\frac{x^2}{1-x} m_\mu^2}$$

[Lautrup,1971]

Muon g-2

Vacuum polarization function vs g-2

Tentative idea to implement (from [Keshavarzi, Marciano, Passera, Sirlin, '20]): Assume the $g-2$ discrepancy can be solely explained by modifying the SM vacuum polarization function contribution.

Problems? Yes, may be in tension with electro-weak precision tests! [Crivellin, Hoferichter, Manzari,Montull, '20], [Malaescu, Schott '21],....

However, [Keshavarzi, Marciano, Passera, Sirlin, '20] suggest that the data for the hadronic cross-section $\sigma(e^+e^- \to \mathit{hadrons})$ may have some missed contributions for $Q \lesssim 0.7$ GeV, energy range in which constraints do not rule out the possibility of explaining the $g-2$ discrepancy.

Muon g-2

Vacuum polarization function vs g-2

AM-Vasquez '21

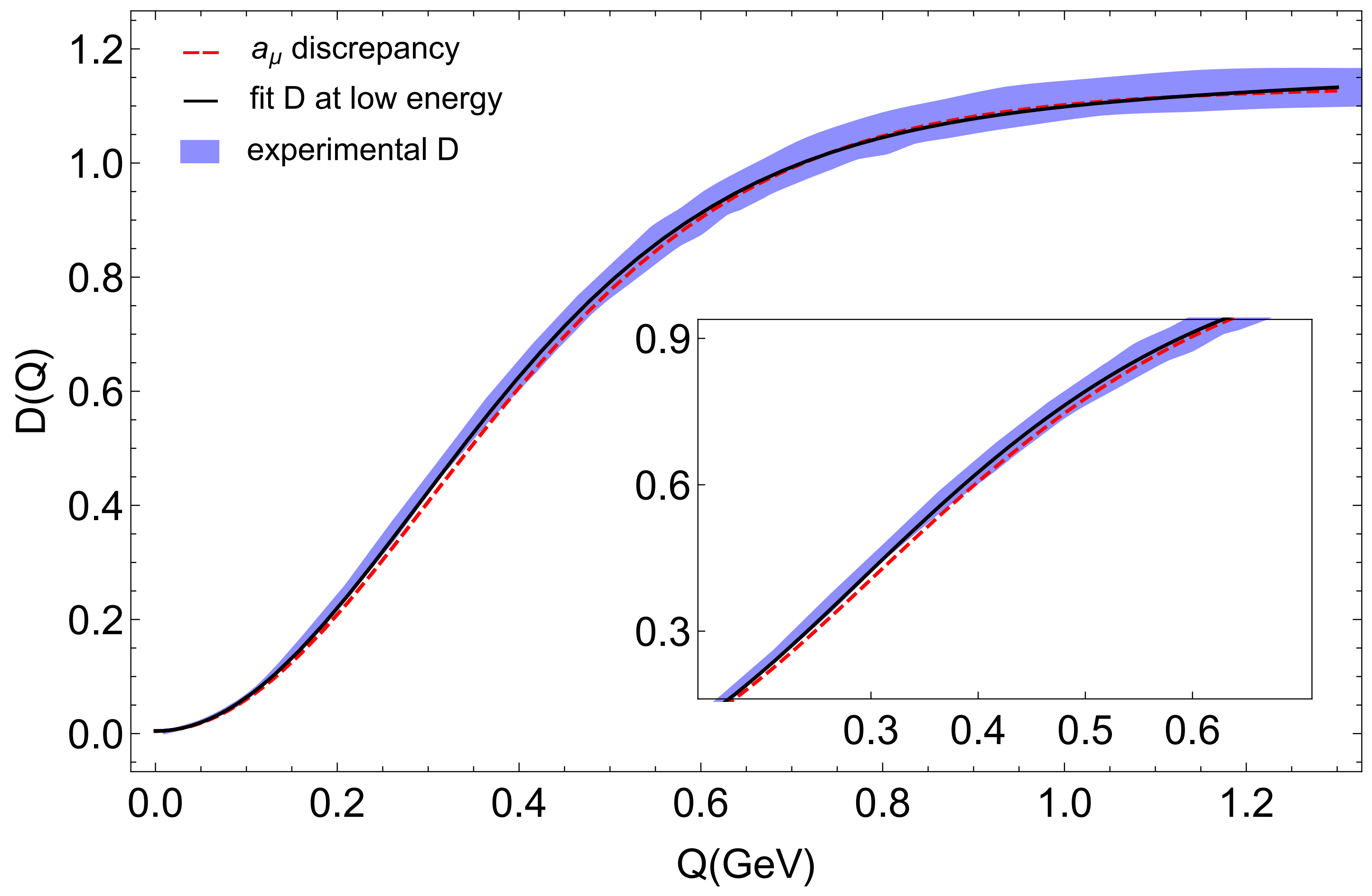


Figure: The Adler function in the energy range $(0, 1.3)$ GeV. The purple region denotes the "experimental" Adler function from tau data. The black line represent the Adler function. For a slightly different value of the constants C, K, c_1, the dashed, red line represents the Adler function saturating the muon $g-2$ discrepancy between experiments and predictions. The inset is a zoom on the region of interest.

$$\alpha_s(Q) = \frac{4\pi}{11 \ln(z+\chi_g) - 2n_f \ln(z+\chi_q)/3},$$

$$z = Q^2/\Lambda^2 \qquad \chi_q = 4m_q^2/\Lambda^2,$$
$$\chi_g = 4m_g^2/\Lambda^2$$

Parameter	Low energy fit	a_μ discrepancy
K	0.80512	0.86501
C	0.23957	0.76396
c_1	-0.35794	-0.18437
Λ	697 MeV	677 MeV

S. Peris, M. Perrottet and E. de Rafael, *Matching long and short distances in large N(c) QCD*, *JHEP* **05**

$$D(Q) = \begin{cases} D_{resurg.}(Q) & Q \leqslant \sqrt{1.6}\ \text{GeV} \\ D_{pert.}(Q) & Q > \sqrt{1.6}\ \text{GeV}\,. \end{cases} \tag{11}$$

Using the values of the low energy fit in Tab. I, we get for the leading contribution of the hadronic vacuum polarization:

$$a_\mu^{(\text{h.v.p.})} = 6.85024 \times 10^{-8}\,. \tag{12}$$

Conclusions

1. We propose a renormalon-based approximation of the QCD Adler function using the Borel-Ecalle resummation procedure of

   O. Costin, Monographs and Surveys in Pure and Applied Mathematics, Chapman and Hall/CRC, 2008.

   merged and applied to the theory of the RGE

   - A. Maiezza and J. C. Vasquez, *Non-local Lagrangians from Renormalons and Analyzable Functions*, *Annals Phys*. 407 (2019) 78–91, [1902.05847].
   - J. Bersini, A. Maiezza and J. C. Vasquez, *Resurgence of the Renormalization Group Equation*, *Annals Phys*. 415 (2020) 168126, [1910.14507].

2. We provide an improvement to perturbation theory and as a result, we get a function that accurately follows the behavior of the data (using an effective running for $\alpha_s(\mu)$)

Conclusions

1. We can reproduce both the leading value for the HVP contribution to a_μ predicted by dispersive approaches, as well as the most recent value consistent with the MUON $g-2$ collaboration data and lattice calculations
2. This opens the possibility of explaining the g-2 anomaly within the SM by including non-analytic corrections in α_s to the VHP contribution

Thank you

Logical Roadmap for the RRGE

$$\Gamma_R^{(2)} \equiv i\left(p^2 - m^2\right) G(L, \alpha_s)$$

- $G(L, \alpha_s) = \gamma_0(\alpha_s) + \sum_{i=1}^{\infty} \gamma_i(\alpha_s) L^i + R(\alpha_s)$, where $R(\alpha_s) \propto n!$

$$\frac{dR(\alpha_s)}{d\alpha_s} = \frac{q}{\beta_0 \alpha_s^2} R(\alpha_s) + \ldots \quad \Longleftrightarrow \quad R(\alpha_S) = \sum_{k=0}^{\infty} C^n R_n(\alpha_s)\, \alpha_s^{k\xi}\, e^{\frac{n}{\beta_0 \alpha_s}}$$

Maiezza, Vasquez

Costin ref.

$+$

$$\left[\dot{\Delta}_\theta, \partial_{\alpha_s}\right] = 0 \Longrightarrow \dot{\Delta}_\theta R(\alpha_s) = A_\theta\, \partial_C R(\alpha_s) \Longrightarrow \dot{\Delta}_\theta R_n(\alpha_s) = (n+1)\, A_\theta\, \alpha_s^{\xi}\, e^{\frac{1}{\beta_0 \alpha_s}}\, R_{n+1}(\alpha_s)$$

Ecalle Ref. | Bridge equation | Resurgence

Backup slides

Operator Product Expansion for Adler function

1. Compare with the usual OPE based transesries structure

$$D\left(Q^2\right) = Q^2\frac{d\Pi\left(Q^2\right)}{dQ^2} = \sum_{k=0} C_k\left(\alpha_s(\mu), \ln\frac{Q^2}{\mu^2}\right) \times \frac{1}{\left(Q^2\right)^k} \times \left\langle \mathcal{O}_k \right\rangle$$

$$= \sum_{k=0}\left[\frac{\left\langle \tilde{\mathcal{O}}_k \right\rangle}{\Lambda^{2k}}\right] \times \left[e^{-\frac{1}{(-\beta_0)\alpha_s(\ell)}}\right]^k \left(-\beta_0\alpha_s(Q)\right)^{k\beta_1/\beta_0^2-\gamma_{0,k}/\beta_0} \times \sum_{n=0} c_k^{(n)}\alpha_s(Q)^n$$

where $\frac{\left\langle \tilde{\mathcal{O}}_k \right\rangle}{\Lambda^{2k}}$ are infinite arbitrary constants related to the resummation prescription

Instead we were able to reduce these infinite arbitrary constants to just one

RGE and renormalons

- The crucial point is that at all orders in perturbation theory

$$\gamma(g) = \gamma_1(g),$$

however this is not true beyond perturbation theory and

$$\gamma(g) - \gamma_1(g) = M(g, R)\,, \text{ where } M(R, g) = q\,R(g) + \frac{1}{2}(rR(g)^2 + 2sg\,R(g))\ldots,$$

and we can write the previous equation as

$$R'(g) = \frac{2q}{\beta_1}\frac{R(g)}{g^2} - \frac{2(\beta_2 q - a\beta_1)}{\beta_1^2}\frac{R(g)}{g} + \mathcal{O}(g^2, g^2 R(g), R(g)^2)\,,$$

Non-perturbative contributions to the anomalous dimension

1. Assume $\beta(g)$ and $\gamma(g)$ are known

2. Then one can in principle solve the RGE to find the desired Green functions

3. We know this is not the whole story since from Renormalons, Green function do have non-perturbative (non-analytic) contributions with arbitrary constants

Non-perturbative contributions to the anomalous dimension

1. Therefore, $\beta(g)$ or $\gamma(g)$ must have non-analytic contributions as well. If fact using the RGE it is possible to show

$$\gamma = \gamma_1 \quad \Leftrightarrow \quad R = 0$$

then there must exist a function $M(R, g)$ such that

$$\gamma = \gamma_1 + M(R(g), g), \qquad M(0,g) = 0$$

Generalized Borel-Laplace resummation: Resurgence (Change here)

- It can be summarized as follows:

1. Given a divergent formal series $y_0(g)$ (solution to the previous equation), one considers the associated formal transseries

$$f(g) = y_0(g) + \sum_{k=1}^{\infty} C^n \, g^{-k\xi} \, e^{-k\eta/g} y_k(g).$$

C is an arbitrary constant, $B(y_0(g))(z)$ has poles at $\eta, 2\eta \, 3\eta, \ldots$ $y_0(g)$ is the function whose asymptotic expansion is identified with perturbation theory

2. For each function $B\left(y_k(g)\right) \equiv Y_k$(z), one builds the functions

$Y_k^{\pm}(z) \equiv Y_k(z \pm i\epsilon)$ (Analytic continuations above or below the real axis)

O. Costin, Monographs and Surveys in Pure and Applied Mathematics, Chapman and Hall/CRC, 2008.

Generalized Borel-Laplace resummation: Resurgence

O. Costin, Monographs and Surveys in Pure and Applied Mathematics, Chapman and Hall/CRC, 2008.

- 3. Resurgence: once $Y_0(z)$ is known, the functions $Y_k(z)$ are given by

$$S_0^k Y_k = \left(Y_0^- - Y_0^{-(k-1)+} \right) \circ \tau_k$$

where

$$Y_k^{-m+} = Y_k^+ + \sum_{j=1}^{m} \binom{k+j}{k} S_0^j Y_{k+j}^+ \circ \tau_{-j} \, .$$

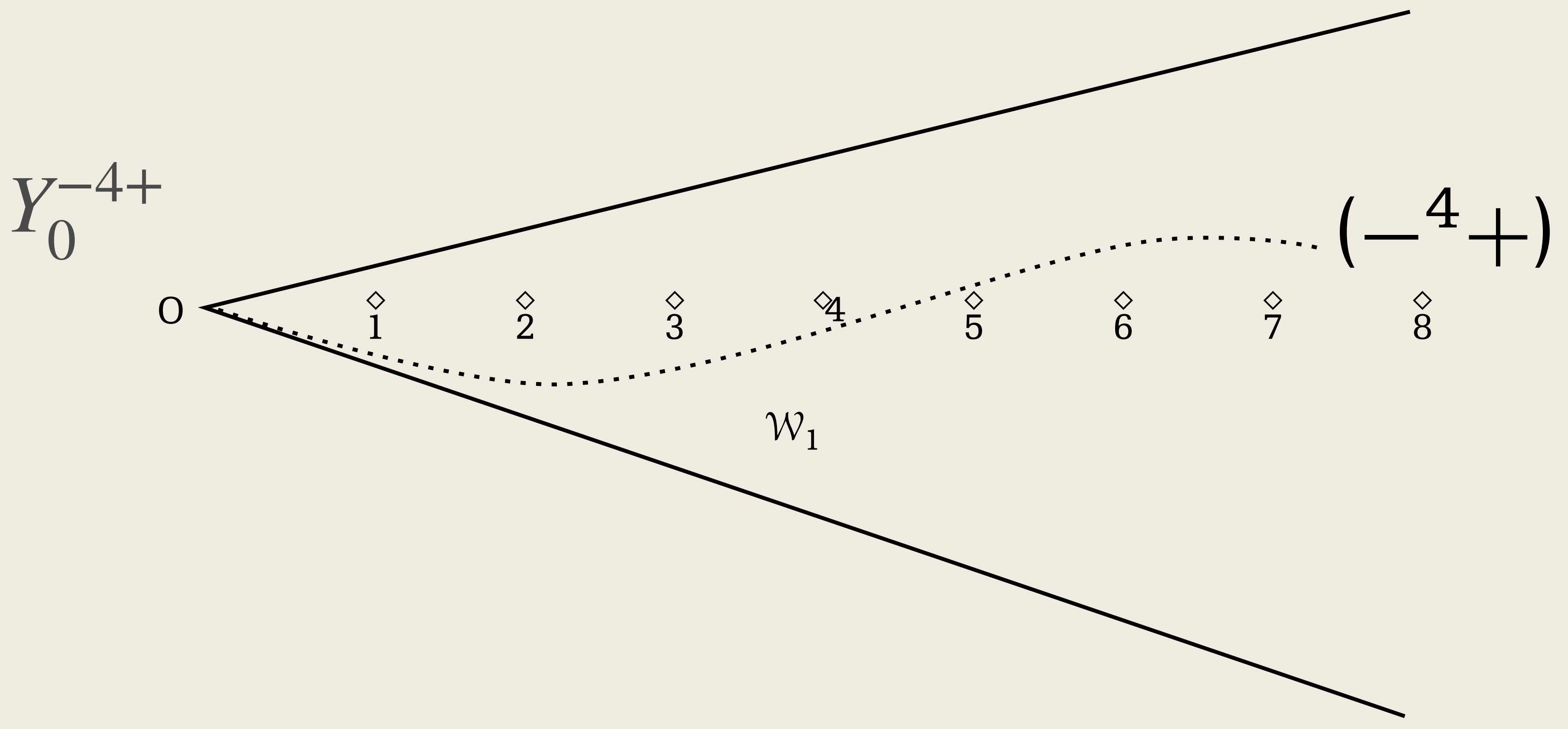


Image taken from Costin 1995

4. The balanced average

$$Y_k^{bal} \equiv Y_k^+ + \sum_{n=1}^{\infty} 2^{-n} \left(Y_k^- - Y_k^{-n-1+} \right).$$

This definition preserves reality in the sense that when $y_0(g)$ is a formal series with real coefficients, then the functions y_k^{bal} are also real $\forall k$. (Costin 2008)

This operation unlike analytic continuation commutes with convolutions.

Generalized Borel-Laplace resummation: Resurgence

- The Laplace transform: when Y_k has poles in the positive real axis, the Laplace transform is modified as follows

$$\mathscr{E}\left(y_k\right) = \mathscr{L} \circ \mathscr{B}\left(y_k\right) = \mathscr{L}\left(Y_k\right) = \int_0^{\infty} Y_k^{bal} e^{-z/g} dz,$$

where the balanced average guaranteed that the reality condition is satisfied

- In the mathematical literature $1/g \to x$, so the asymptotic expansions when $x \to \infty$ correspond to the weak coupling limit $g \to 0$.

O. Costin, Monographs and Surveys in Pure and Applied Mathematics, Chapman and Hall/CRC, 2008.

Example: the simplest equation

$-g^2 y'(g) + y = g$, the solution is of the form $y(x) = \sum_n a_n n! x^n$ so it is divergent

(Adding non linear terms g^n give a infinite number of singularities in the Borel transform of the solution)

The Borel transform $\mathscr{B}(y(g)) \equiv Y_0(z)$ is

$$Y_0(z) = \frac{1}{1-z}.$$

1. Write the formal solution

$$y(g) = y_0(g) + \sum_{k=1}^{\infty} C^k e^{-k/g} y_k(g),$$

2. Build the analytic continuations $Y_0^{\pm}(z) = \dfrac{1}{1-(z \pm i\epsilon)}$

3. Resurgence property: $S^k Y_k = \left(Y_0^- - Y_0^{-k-1+}\right) \circ \tau_k; \quad \tau_k : z \to z + k.$

3.1. Let's elaborate on the non-perturbative functions $Y_k(z),\ k \geq 1$, using the resurgence property

$$\left(Y_0^- - Y_0^{-0+}\right) \circ \tau_1 = SY_1(z)$$

$Y^{-0+} = Y^+(z),$ then

$$SY_1(z) = \left(Y_0^- - Y_0^+\right) \circ \tau_1 = -2\pi i \delta(1-z) \circ \tau_1 = -2\pi i \delta(z)$$

$$Y_1 = -\frac{2\pi i}{S}\delta(z)$$

3.2. $S^2 Y_2(z) = (Y_0^- - Y_0^{-1+}) \circ \tau_2,$

$Y_0^{-1+} = Y^+ + SY_1^+ \circ \tau_{-1}$, so

$$S^2 Y_2(z) = [Y_0^+ - Y_0^- - SY_1^+ \circ \tau_{-1}] \circ \tau_2$$

$S^2 Y_2(z) = [SY_1 \circ \tau_{-1} - SY_1^+ \circ \tau_{-1}] \circ \tau_2$ = 0.

The same applies to $Y_2(z) = Y_3(z) = \ldots = 0$.

3.3. The balanced average for $Y_0(z)$ and $Y_1(z)$:

$$Y_k^{bal} \equiv Y_k^+ + \sum_{n=1}^{\infty} 2^{-n}(Y_k^- - Y_k^{-n-1+}),$$

expanding

$$Y_0^{bal} = Y_0^+ + \frac{1}{2}(Y_0^- - Y_0^{-0+}) + \frac{1}{2^2}(Y_0^- - Y_0^{-1+}) + \ldots$$

i) $Y_0^- - Y_0^{-0+} = Y_0^- - Y_0^+$,
ii) $Y_0^- - Y_0^{-1+} = Y_0^- - Y_0^+ - SY_1^+ \circ \tau_{-1} = Y_0^- - Y_0^+ - (Y_0^- - Y_0^+) \circ \tau_1 \circ \tau_{-1} = 0$

In the same way and using that $Y_2(z) = Y_3(z) = \ldots = 0$, the other terms also vanishes and

$$Y_0^{bal} = \frac{1}{2}(Y_0^+ + Y_0^-),$$

which give precisely the P.V. of the Laplace integral.

In the same way it can be shown that

$$Y_1^{bal}(z) = \frac{1}{2}(Y_1^+ + Y_1^-)$$

and the solution is given by

$$y(g) \mapsto \sigma(y(g)) = e^{-1/g}\mathsf{Ei}\left(1/g\right) - \frac{4\pi^2 C}{S} e^{-1/g},$$

which is the well known solution that can be found by other methods.

The sum of a Borel-Écalle summable transseries is by definition an analyzable function

The Borel-Ecalle Resummation procedure

Image taken from O. Costin, Monographs and Surveys in Pure and Applied Mathematics, Chapman and Hall/CRC, 2008.

$$\begin{tikzcd} \boxed{\text{Convergent series}} \arrow[r] \arrow[d] & \boxed{\text{Summation}} \arrow[r] & \boxed{\text{Analytic functions}} \arrow[d] \\ \boxed{\text{Closure under ``all'' operations}} \arrow[d] & & \boxed{\text{Closure under ``all'' operations}} \arrow[d] \\ \boxed{\text{Transseries}} \arrow[r] & \boxed{\text{E-B Summation}} \arrow[r] & \boxed{\text{Analyzable functions}} \end{tikzcd}$$

This is the only known way to close functions under the listed operations.

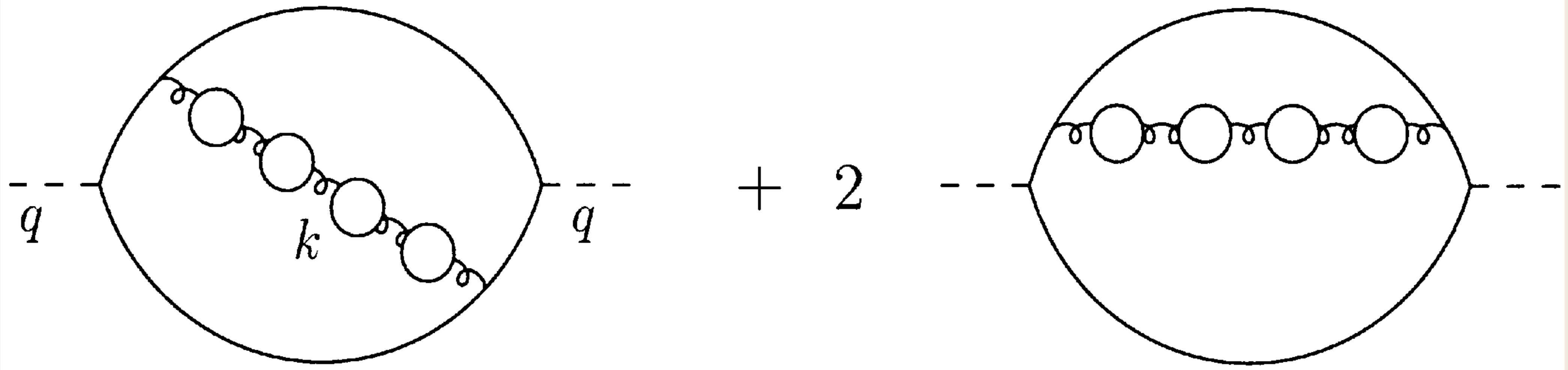
The resummation of a transseries is by definition an Analyzable Function

$$\sum_{n}^{\infty} C^n y_n(x) e^{-n\lambda x} \quad \rightarrow \text{Borel-Ecalle summation} \rightarrow \text{Analyzable Function}$$

I will only discuss one-parameter transseries relevant to renormalons at the "leading" order

The Adler function and Resurgence

1. On to of the perturbative result. We consider the fermion-bubble contributions



These contributions go as $n!$

(D.J. Broadhurst, Z. Phys. C 58 (1993) 339–346, https://doi.org/10.1007/BF01560355.)

and the Borel transform goes as

$$\frac{1}{K}B[D_{bubble}](u) = \sum_{n=0} \frac{d_n}{n!}u^n = \frac{32}{3}\left(\frac{Q^2}{\mu^2}e^C\right)^{-u} \frac{u}{1-(1-u)^2} \sum_{k=2}^{\infty} \frac{(-1)^k k}{\left(k^2-(1-u)^2\right)^2},$$

The Adler function and Resurgence

1. We rewrite it as ($\mu^2 = Q^2 e^{-5/3}$.)

(M. Neubert, Phys. Rev. D 51 (1995) 5924–5941, https://doi.org/10.1103/PhysRevD.51.5924, arXiv: hep-ph/9412265.)

$$\frac{1}{K\,C_F} B[D_{bubble}](z) \to \frac{3e^{10/3}\mu^4}{2\beta_0 Q^4\left(\frac{2}{\beta_0}+z\right)} - \frac{e^5\mu^6}{\beta_0^2 Q^6\left(\frac{3}{\beta_0}+z\right)^2} -$$

$$\sum_{p=1}^{\infty}\left[\frac{\mu^4 e^{\frac{10(p+1)}{3}}\left(\frac{Q}{\mu}\right)^{-4p}}{\beta_0^2 p(2p+1)Q^4\left(\frac{2p+2}{\beta_0}+z\right)^2} - \frac{\mu^6 e^{\frac{10p}{3}+5}\left(\frac{Q}{\mu}\right)^{-4p}}{\beta_0^2(p+1)(2p+1)Q^6\left(\frac{2p+3}{\beta_0}+z\right)^2}\right].$$

Such that the pole structure of the Borel transform is manifest