

Correlated Dirac Eigenvalues and Axial Anomaly in Chiral Symmetric QCD

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based on PRL 126 (2021) 082001, arXiv:2010.14836

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Outline

- Motivation
- $\partial^n \rho / \partial^n m_l$ & C_{n+1} and $U_A(1)$ symmetry
- Lattice Setup
- Results
- Summary & Conclusions

Symmetries of QCD

$$\mathcal{L} = -\frac{1}{4}F_a^{\mu\nu}F_{\mu\nu}^a + \sum_{q \in u,d,s,c,b,t} \bar{q}[i\gamma^\mu(\partial_\mu - igA_\mu) - m_q]q$$

$$SU_L(N_f) \times SU_R(N_f) \times U_V(1) \times \boxed{U_A(1)} \quad (m_q = 0)$$

★ $\boxed{SU_L(N_f) \times SU_R(N_f)}$ chiral symmetry

- ⌚ SSB in the vacuum: $SU_L(N_f) \times SU_R(N_f) \rightarrow SU_V(N_f)$
- ⌚ Restored at $T \geq T_c$

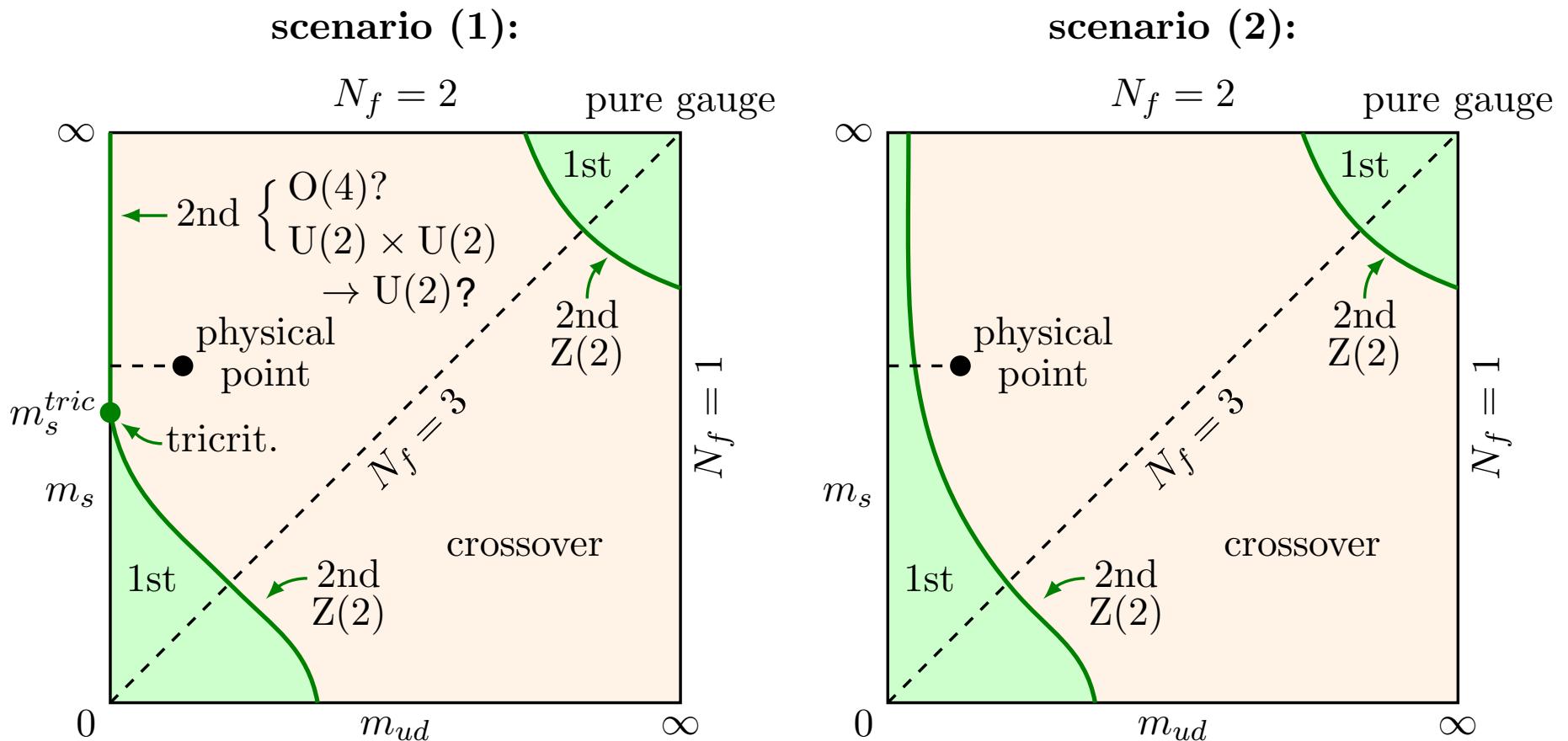
★ $\boxed{U_A(1)}$ symmetry

- ⌚ Broken on the quantum level due to ABJ anomaly

$$\partial_\mu j_5^\mu = \frac{g^2 N_f}{16\pi^2} F_a^{\mu\nu} \tilde{F}_{\mu\nu}^a \neq 0 \quad (\tilde{F}_{\mu\nu}^a \equiv \frac{1}{2}\epsilon_{\mu\nu\lambda\rho} F_a^{\lambda\rho})$$

$U_A(1)$ symmetry & Chiral phase transition

The nature of chiral phase transition depends on
how the axial anomaly manifests itself at $T \sim T_c$



Pisarski, Wilczek PRD 29 (1984) 338

Butti et. al., JHEP 08 (2003) 029

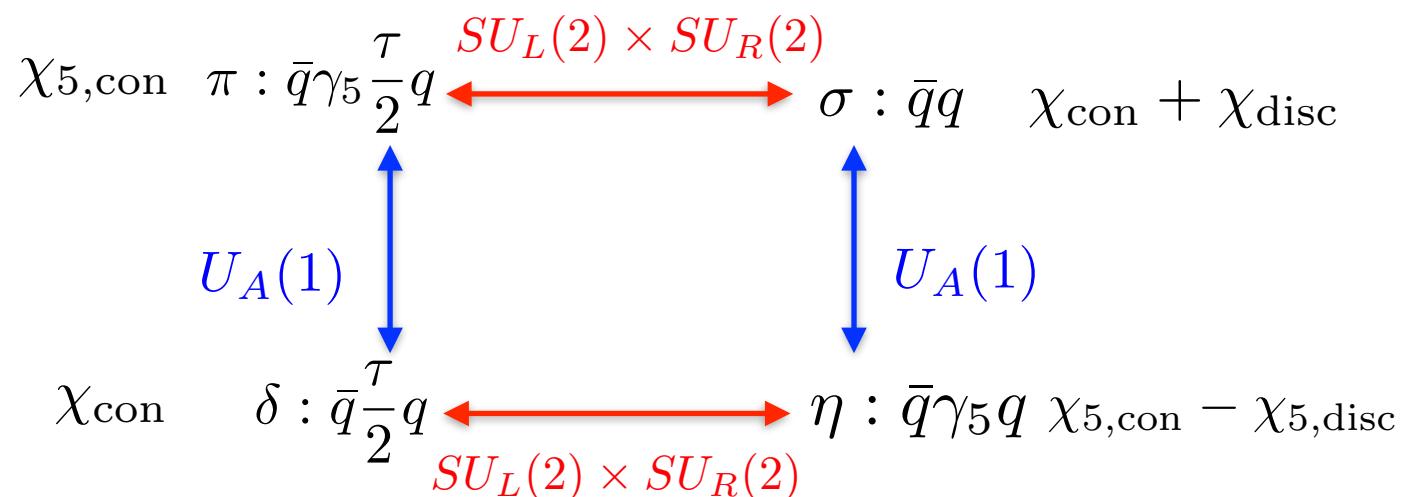
Pelissetto & Vicari, PRD 88 (2013) 105018

Grahl, PRD 90 (2014) 117904

Signatures of symmetry restorations

- Susceptibilities defined as integrated two point correlation functions of the local operators, e.g. $\chi_\pi = \int d^4x \langle \pi^i(x) \pi^i(0) \rangle$

HotQCD PRD 86 (2012) 094503



$$SU_L(2) \times SU_R(2)$$

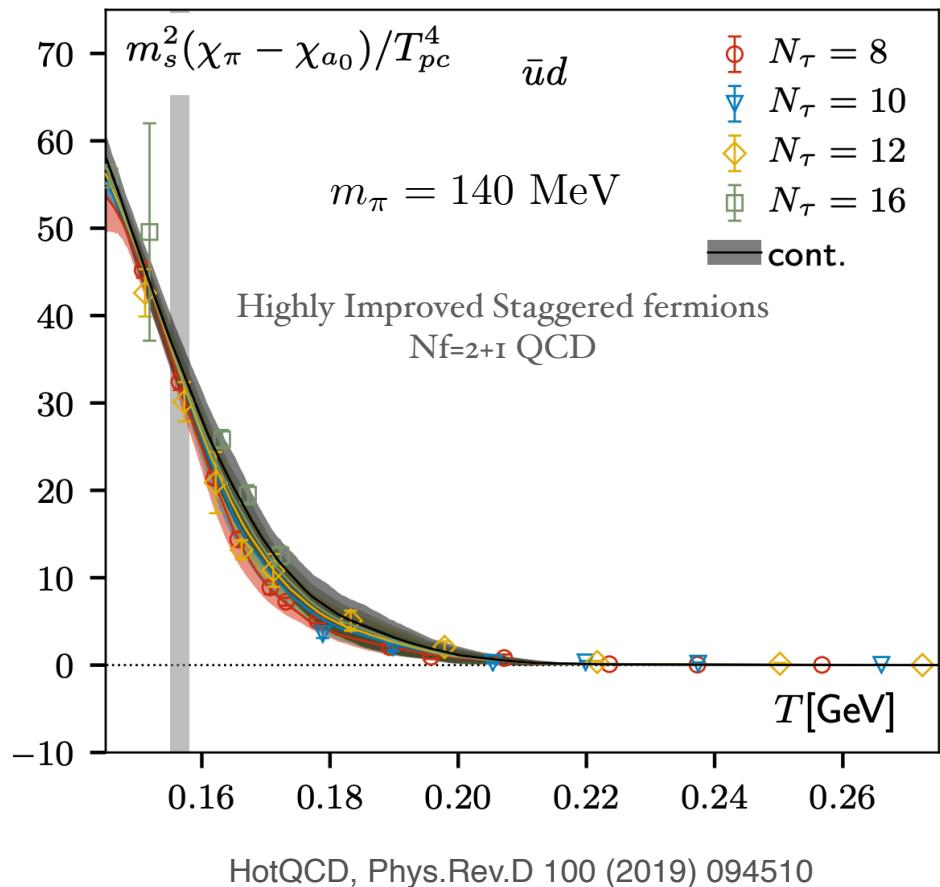
$$\chi_\pi - \chi_\delta = \chi_{\text{disc}}$$

$$U_A(1)$$

$$\chi_\pi - \chi_\delta = \chi_{\text{disc}} = 0$$

$$\chi_{\text{disc}} = \frac{T}{V} \int d^4x \langle [\bar{\psi}(x)\psi(x) - \langle \bar{\psi}(x)\psi(x) \rangle]^2 \rangle$$

Status of lattice studies on axial anomaly

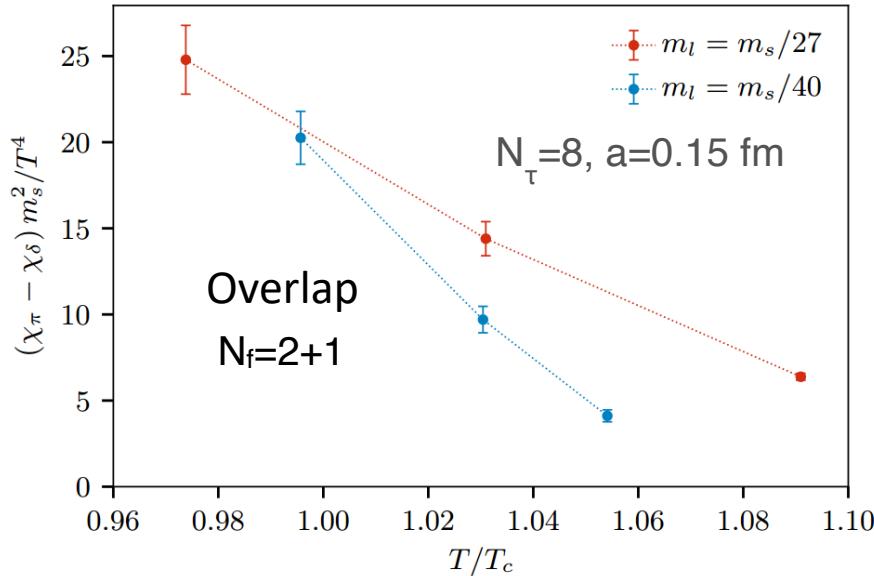


- At $T \leq T_{pc}$ for physical pion mass axial anomaly remains manifested in $\chi_\pi - \chi_\delta$

See similar conclusions obtained using chiral fermions:
HotQCD, PRL 113(2014) 082001, PRD 89 (2014) 054514
JLQCD, arXiv:2011.01499,...

- What happens in the chiral limit?

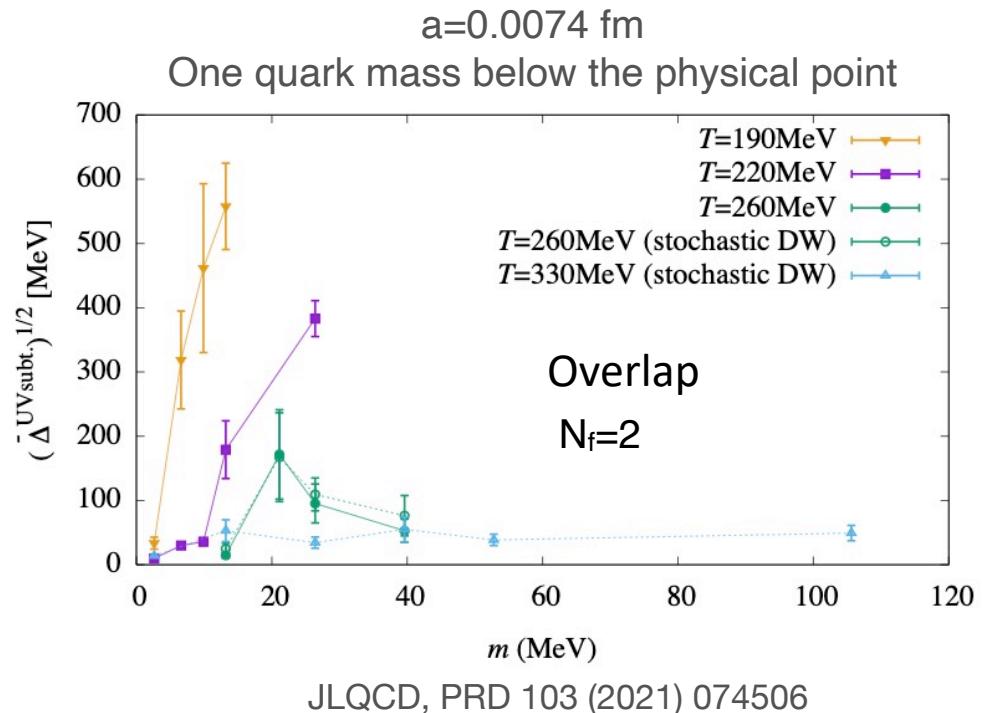
Status of lattice studies on Axial anomaly



L. Mazur et al., arXiv:1811.08222

Remains manifested for
 $m_\pi = 110 \text{ MeV}$ at $T < 1.1 T_c$

See similar conclusions from
Ohno et al., PoS Lattice 2012 (2012) 095,
Dick et al., PRD 91(2015) 094504,...



JLQCD, PRD 103 (2021) 074506

Seems to disappear at $T \geq 220 \text{ MeV}$

See similar conclusions from
Chiu et al., PoS Lattice 2013 (2014) 165,
Tomiya et al., [JLQCD] PRD 96 (2017) 034509,...

The fate of $U_A(1)$ still unsettled due to
the lack of continuum and chiral extrapolations

Signatures of symmetry restorations in ρ

$$\langle \bar{\psi} \psi \rangle = \int_0^\infty d\lambda \frac{4m_l \rho}{\lambda^2 + m_l^2}$$

$$\chi_\pi - \chi_\delta = \int_0^\infty d\lambda \frac{8m_l^2 \rho}{(\lambda^2 + m_l^2)^2}$$

• Restoration of $SU_L(2) \times SU_R(2)$ symmetry :

- ✿ $\rho(0) = 0$ as from Banks-casher relation: $\lim_{m_l \rightarrow 0} \langle \bar{\psi} \psi \rangle = \pi \rho(0)$

Banks and Casher,
NPB 169 (1980) 103

- ✿ Partition function is an even function in m_l due to the $Z(2)$ subgroup

• Effective restoration of $U_A(1)$ symmetry :

- ✿ A sizable gap in the near-zero modes, i.e. $\rho(\lambda < \lambda_c) = 0$

Cohen, nucl-th/980106

- ✿ If ρ is analytic in m_l^2 , $U_A(1)$ breaking is absent in up to 6 point correlation functions of π and δ

Aoki, Fukaya and Taniguchi, PRD 86 (2012) 114512

Possible behaviors of ρ making $SU_L(2) \times SU_R(2)$ restored but not $U_A(1)$

- ➊ Dilute instanton gas approximation $\rho \sim m^2 \delta(\lambda)$ will lead to $U_A(1)$ breaking even in the chiral limit

Gross, Yaffe & Pisarski, RMP 81'

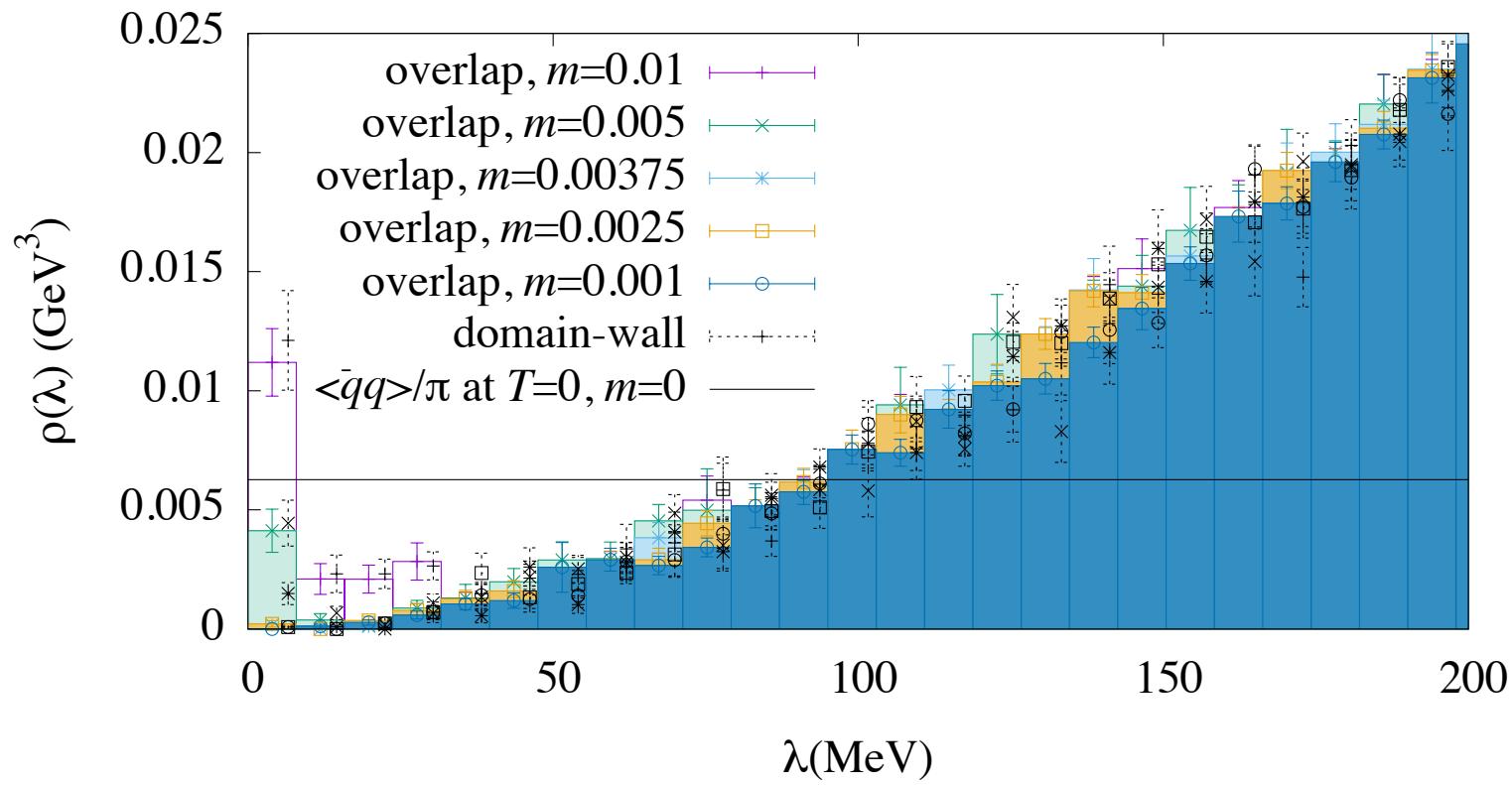
- ➋ LQCD: At high T for the physical m_l , the T dependence of χ_t follows dilute instanton gas approximation prediction

See a recent review, Lombardo & Trunin,
IJMPA 35 (2020) 2030010

Due to $\rho \sim m^2 \delta(\lambda)$? what happens for $m_l \rightarrow 0$?

Microscopic origin in ρ

$\beta=4.30, T=220\text{MeV}, L=32(2.4\text{fm})$



JLQCD, PRD 103 (2021) 074506

- No clear gap
- As m_l gets smaller, the infrared enhancement seems disappeared, at $m_l < 0.01$ mass dependence can be hardly seen

Novel relation: quark mass derivatives of ρ & C_2

Eigenvalue spectrum for (2+1)-flavor QCD:

$$\rho(\lambda, m_l) = \frac{T}{VZ[U]} \int D[U] e^{-S_G[U]} \det[\not{D}[U] + m_s] \times (\det[\not{D}[U] + m_l])^2 \rho_U(\lambda)$$

Eigenvalue spectrum for a given configuration: $\rho_U(\lambda) = \sum_j \delta(\lambda - \lambda_j)$

Partition function: $Z[U] = \int D[U] e^{-S_G[U]} \det[\not{D}[U] + m_s] \times (\det[\not{D}[U] + m_l])^2$

$$\det[\not{D}[U] + m_l] = \prod_j (+i\lambda_j + m_l)(-i\lambda_j + m_l) = \exp \left(\int_0^\infty d\lambda \rho_U(\lambda) \ln[\lambda^2 + m_l^2] \right)$$



$$\frac{V}{T} \frac{\partial \rho}{m_l} = \int_0^\infty d\lambda_2 \frac{4m_l C_2}{\lambda_2^2 + m_l^2}$$

$$C_2(\lambda, \lambda_2) = \langle \rho_U(\lambda) \rho_U(\lambda_2) \rangle - \langle \rho_U(\lambda) \rangle \langle \rho_U(\lambda_2) \rangle$$

Novel relation: $\partial^n \rho / \partial^n m_l$ & C_{n+1}

$$\frac{V}{T} \frac{\partial^2 \rho}{m_l^2} = \int_0^\infty d\lambda_2 \frac{4(\lambda_2^2 - m_l^2) C_2}{(\lambda_2^2 + m_l^2)^2} + \int_0^\infty d\lambda_2 d\lambda_3 \frac{(4m_l)^2 C_3}{(\lambda_2^2 + m_l^2)(\lambda_3^2 + m_l^2)}$$

... ...

... ...

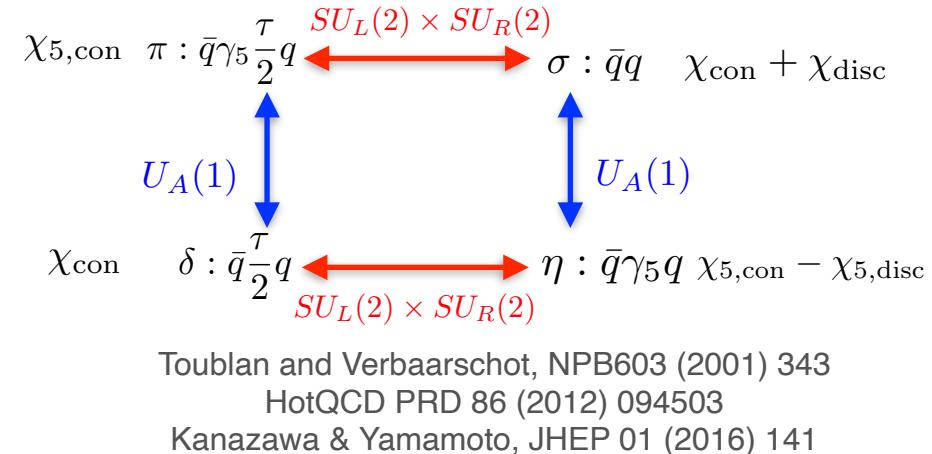
$$C_n(\lambda_1, \dots, \lambda_n; m_l) = \left\langle \prod_{i=1}^n [\rho_U(\lambda_i) - \langle \rho_U(\lambda_i) \rangle] \right\rangle$$

Signatures of symmetry restorations in ρ

Chiral symmetry restoration: $\chi_\pi - \chi_\delta = \chi_{\text{disc}}$

$$\chi_\pi - \chi_\delta = \int_0^\infty d\lambda \frac{8m_l^2 \rho}{(\lambda^2 + m_l^2)^2}$$

$$\chi_{\text{disc}} = \int_0^\infty d\lambda \frac{4m_l \partial\rho/\partial m_l}{\lambda^2 + m_l^2}$$



If eigenvalues are uncorrelated, they obey the Poisson statistics:

$$C_n^{\text{Po}}(\lambda_1, \dots, \lambda_n) = \delta(\lambda_1 - \lambda_2) \dots \delta(\lambda_n - \lambda_{n-1}) \langle \rho_U(\lambda_1) \rangle + \mathcal{O}(1/N)$$

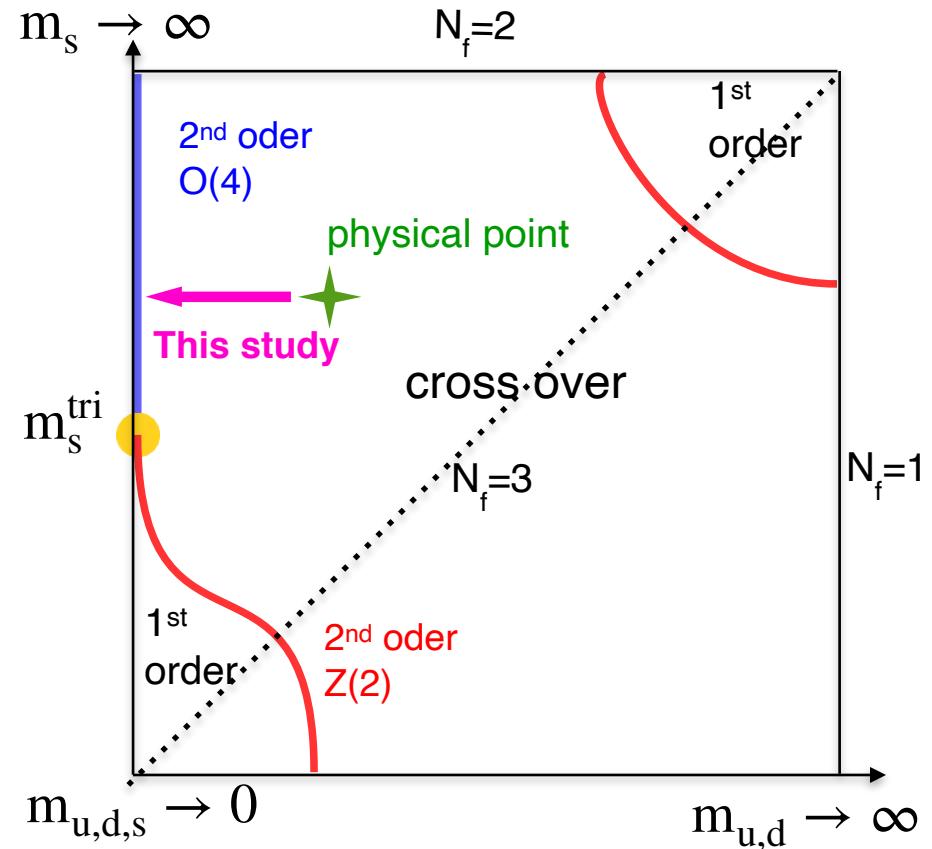
$$\left(\frac{\partial \rho}{\partial m_l} \right)^{\text{Po}} = \frac{4m_l \rho}{\lambda^2 + m_l^2} - \frac{V \rho}{TN} \langle \bar{\psi} \psi \rangle \quad \xrightarrow{\textcolor{blue}{\longrightarrow}} \quad \chi_{\text{disc}}^{\text{Po}} = 2(\chi_\pi - \chi_\delta)$$

Non-Poisson correlation among eigenvalues are needed for chiral symmetry restoration if $\chi_\pi - \chi_\delta \neq 0$

Kanazawa & Yamamoto,
JHEP 01 (2016) 141

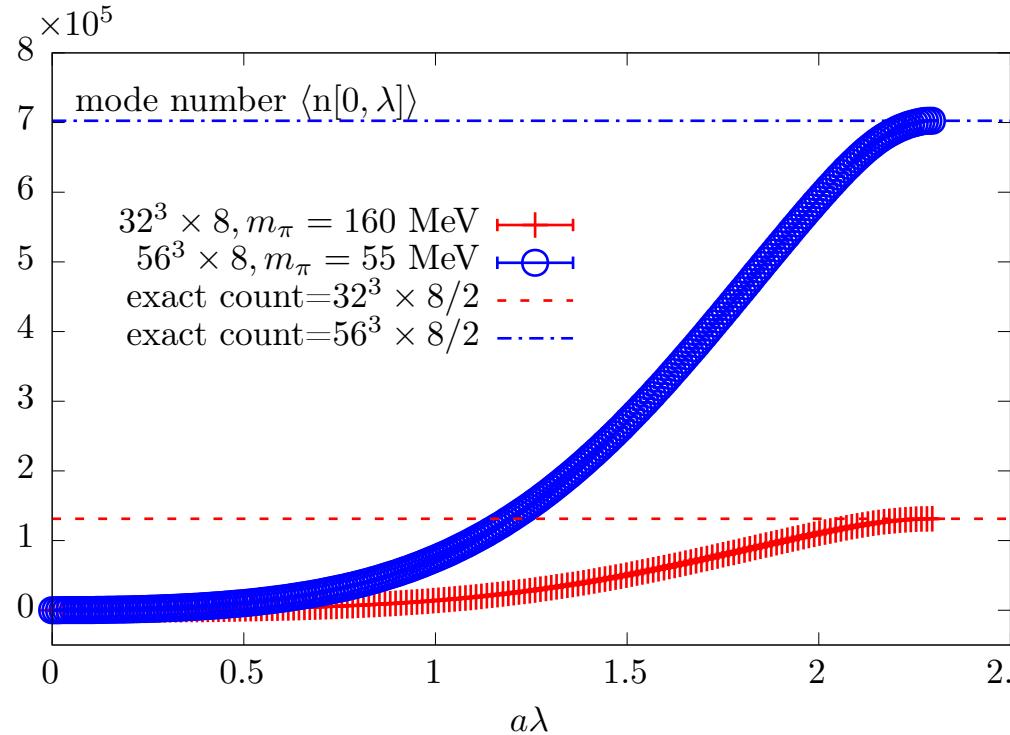
Lattice Setup

- At $T \sim 205$ MeV ($1.6 T_c$)
- HISQ/tree action
- $N_f = 2+1$:
 - $N_\tau = 8, 12, 16$ ($a = 0.12, 0.08, 0.06$ fm)
 - $m_s^{\text{phy}}/m_l = 20, 27, 40, 80, 160$
($m_\pi = 160, 140, 110, 80, 55$ MeV)
 - $4 \leq N_\sigma/N_\tau \leq 9$
- ρ is obtained from the Chebyshev filtering technique

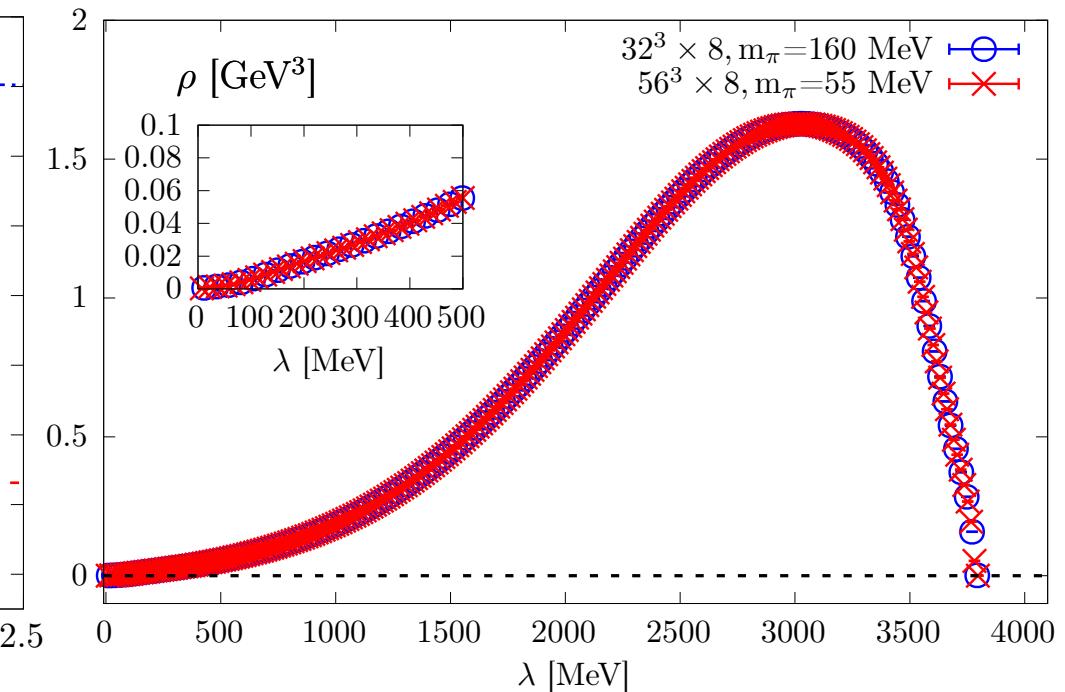


Nuclear Science
Computing Center at CCNU

Mode number and Complete ρ



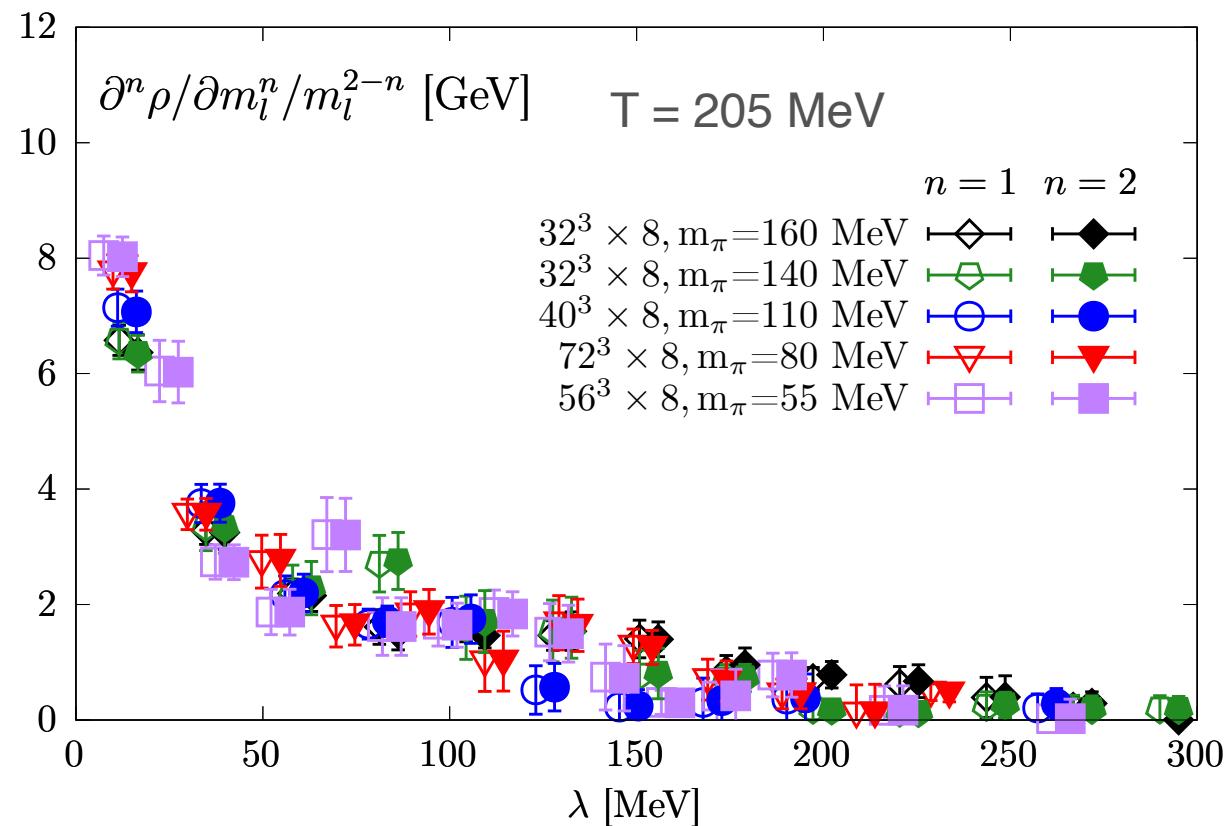
📍 Converges to the exact count



📍 Mass dependence can be
hardly observed from ρ directly

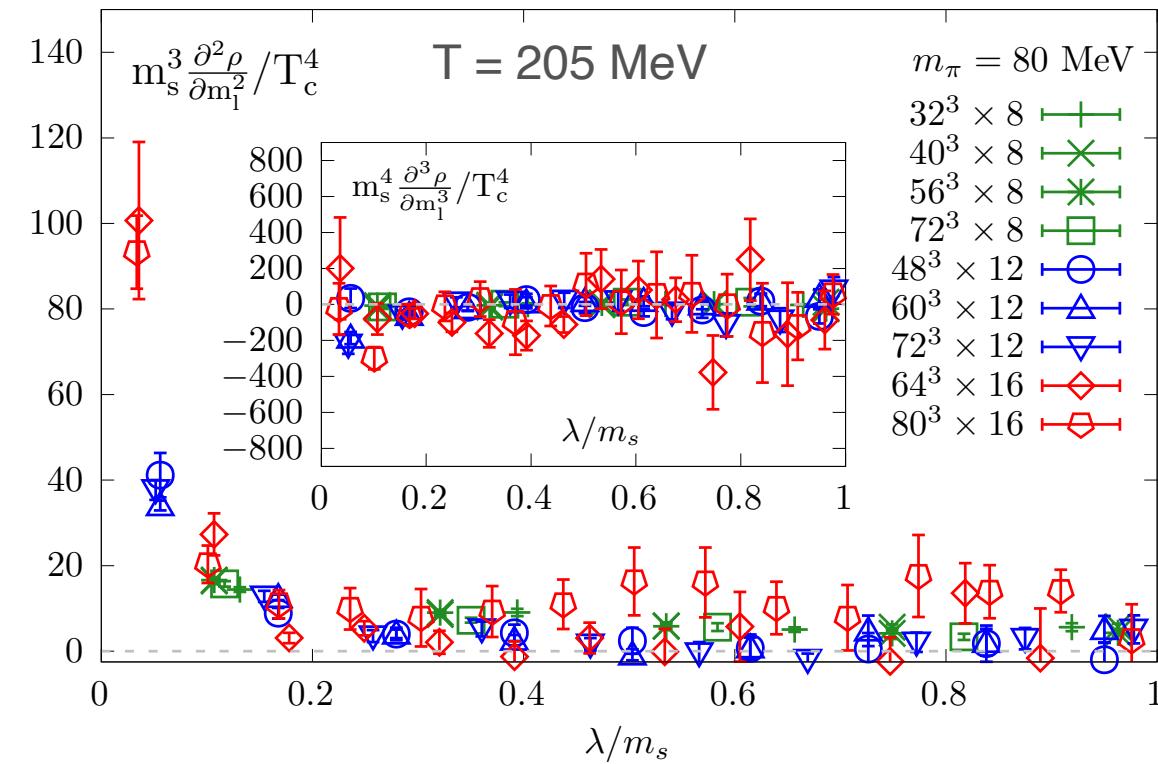
Utilize the Chebyshev filtering technique combined with
a stochastic estimate of the mode number

1st and 2nd mass derivatives of ρ on $N_\tau=8$ lattices



- $m_l^{-1}(\partial\rho/\partial m_l) \approx \partial^2\rho/\partial m_l^2$
- Quark mass independent
- Peaked structure develops at $\lambda \rightarrow 0$ and drops rapidly towards zero for $\lambda/T > 1$

2nd and 3rd mass derivatives of ρ : volume and a dependences



- Peaked structure becomes sharper towards continuum limit

- Mild volume dependence

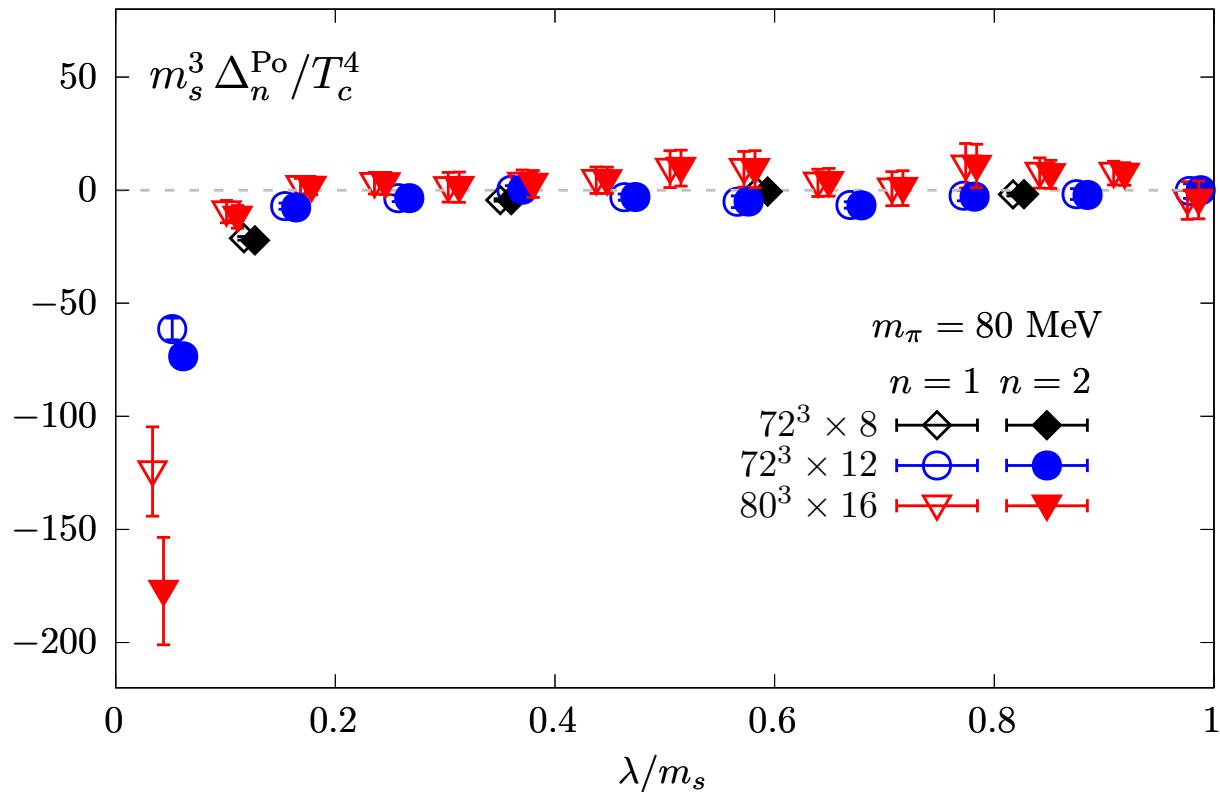
$$\frac{\partial^3 \rho}{\partial m_l^3} \approx 0$$

$$m_l^{-1} \left(\frac{\partial \rho}{\partial m_l} \right) \approx \frac{\partial^2 \rho}{\partial m_l^2}$$

$$\rho(\lambda \rightarrow 0, m_l \rightarrow 0) \propto m_l^2$$

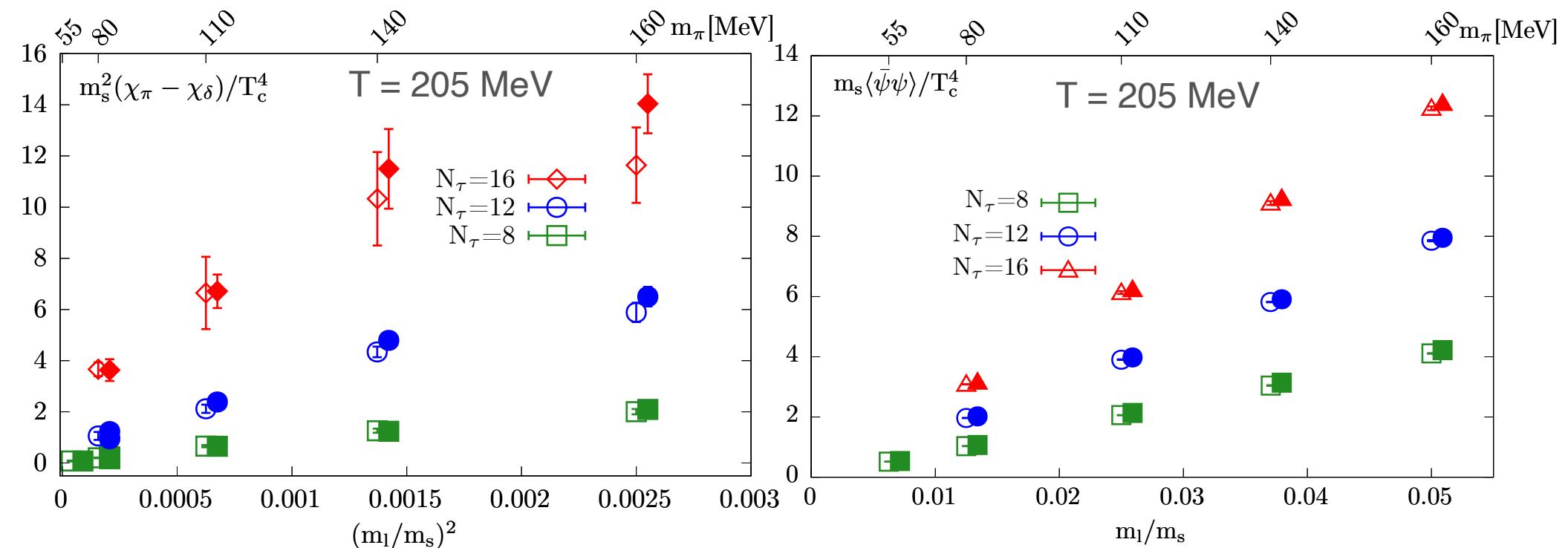
Non-Poisson correlations

$$\Delta_n^{\text{Po}} = m_l^{n-2} [\partial^n \rho / \partial m_l^n - (\partial^n \rho / \partial m_l^n)^{\text{Po}}]$$



Repulsive non-Poisson correlation at small λ range gives rise to the $\rho(\lambda \rightarrow 0)$ peak

Quantities related to ρ



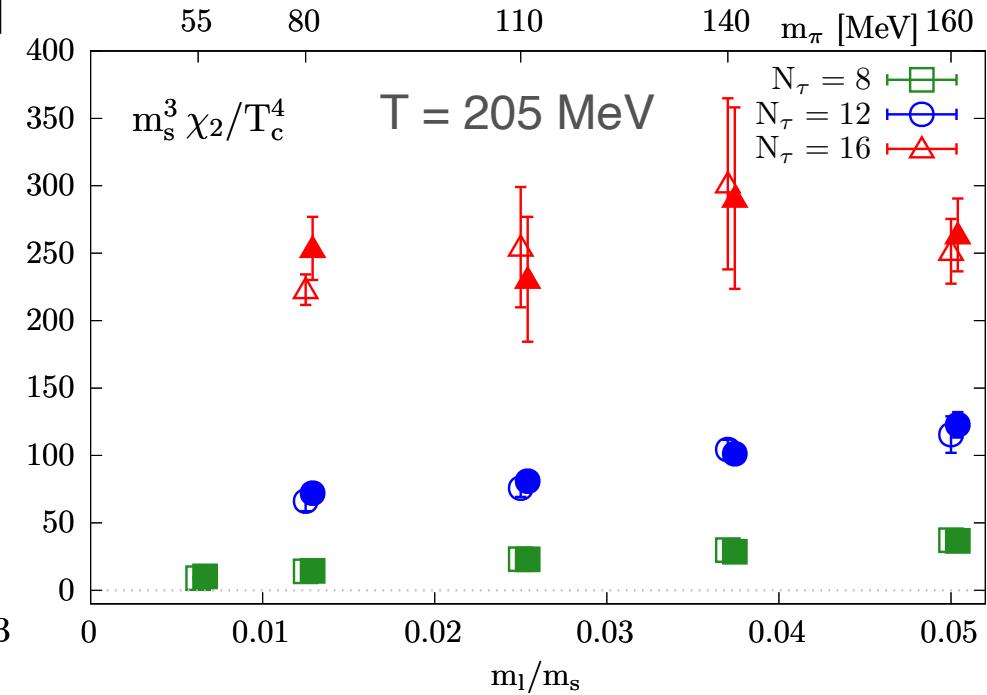
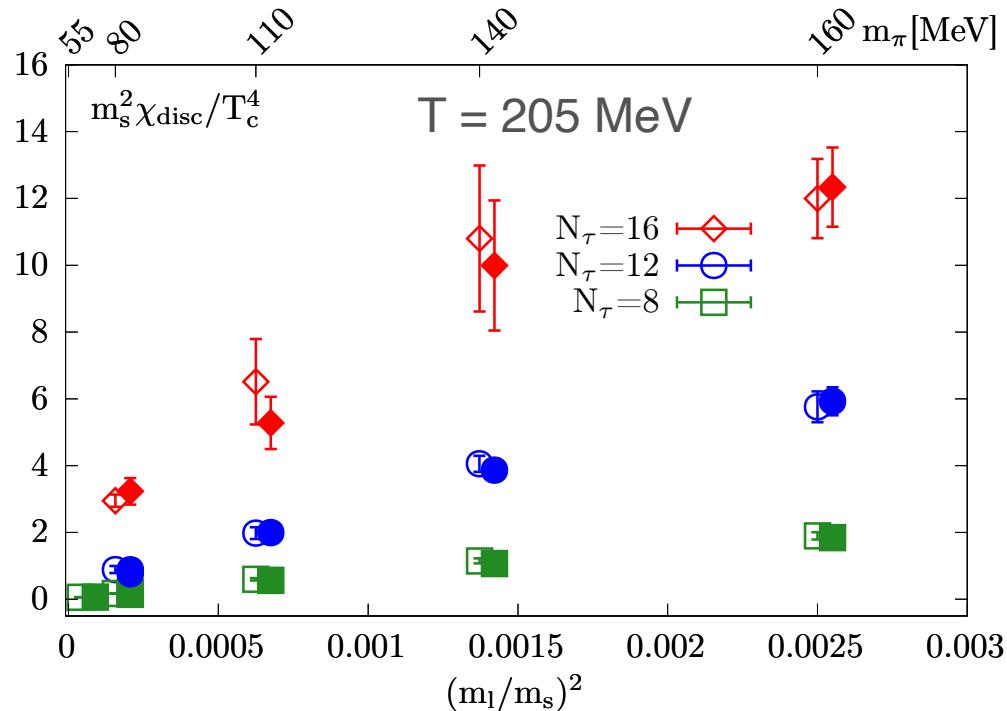
Open symbol: Results obtained from the fermion matrix inversion
 Filled symbol: reconstructed from ρ

$$\chi_\pi - \chi_\delta = \int_0^\infty d\lambda \frac{8m_l^2 \rho}{(\lambda^2 + m_l^2)^2}$$

$$\langle \bar{\psi} \psi \rangle = \int_0^\infty d\lambda \frac{4m_l \rho}{\lambda^2 + m_l^2}$$

$\langle \bar{\psi} \psi \rangle$ is reproduced very well from ρ

Quantities related to 1st and 2nd mass derivatives of ρ

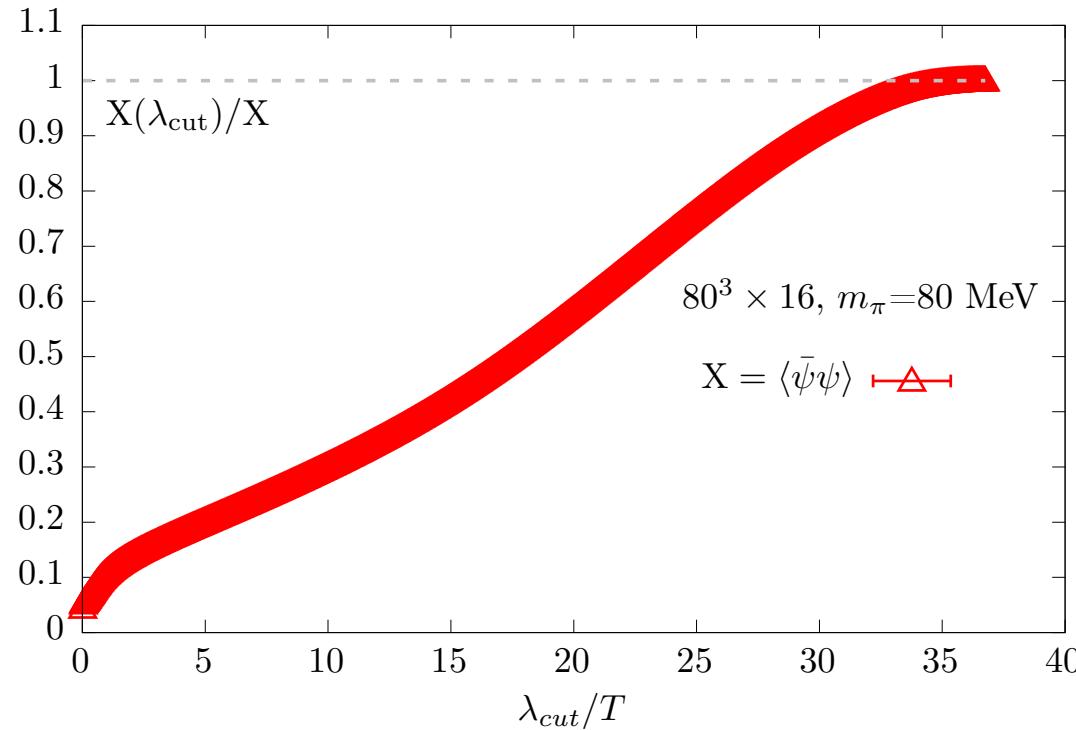


$$\chi_{disc} = \int_0^\infty d\lambda \frac{4m_l \partial \rho / \partial m_l}{\lambda^2 + m_l^2}$$

$$\chi_2 = \int_0^\infty d\lambda \frac{4m_l \partial^2 \rho / \partial m_l^2}{\lambda^2 + m_l^2}$$

χ_{disc} and χ_2 are successfully reproduced from the 1st and 2nd mass derivatives of ρ

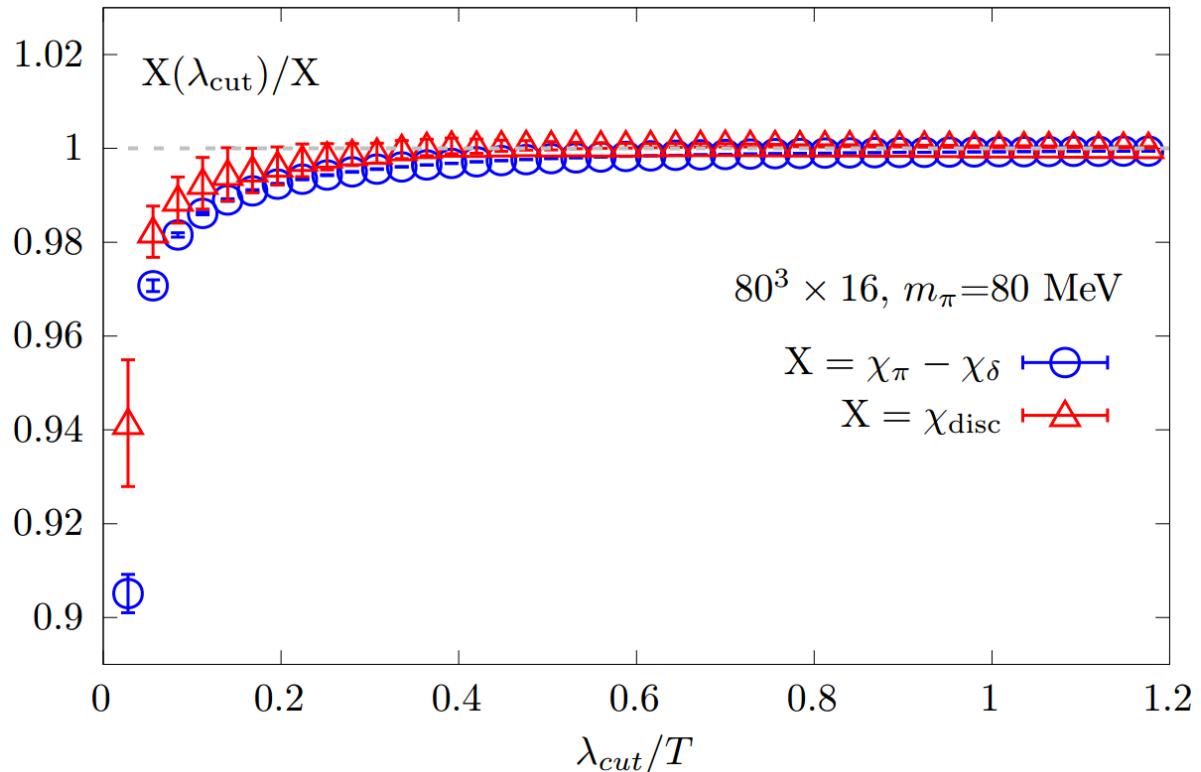
UV divergence of chiral condensate



$$\langle\bar{\psi}\psi\rangle(\lambda_{cut}) = \int_0^{\lambda_{cut}} d\lambda \frac{4m_l\rho}{\lambda^2 + m_l^2}$$

Full ρ is needed for reproduction of chiral condensate

Infrared contribution to two $U_A(1)$ measures

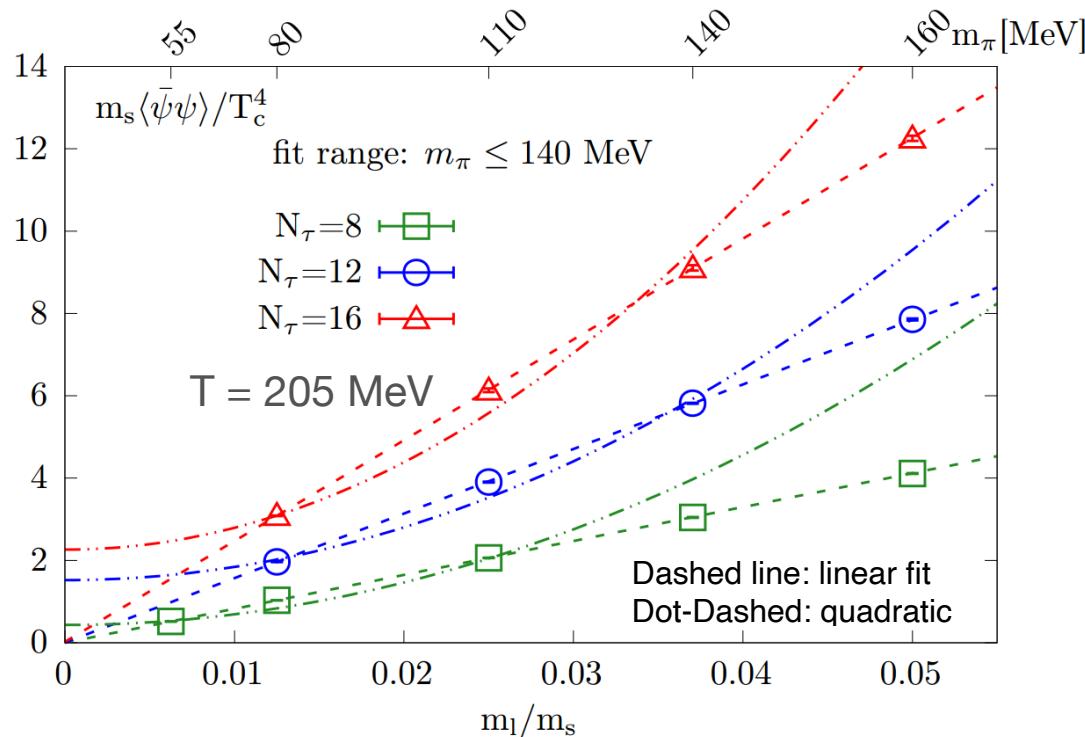


$$(\chi_\pi - \chi_\delta)(\lambda_{\text{cut}}) = \int_0^{\lambda_{\text{cut}}} d\lambda \frac{8m_l^2 \rho}{(\lambda^2 + m_l^2)^2}$$

$$\chi_{\text{disc}}(\lambda_{\text{cut}}) = \int_0^{\lambda_{\text{cut}}} d\lambda \frac{4m_l \partial \rho / \partial m_l}{\lambda^2 + m_l^2}$$

Only infrared part of ρ and $\partial \rho / \partial m_l$ are needed for the reproduction

$SU_L(2) \times SU_R(2)$ symmetry restoration



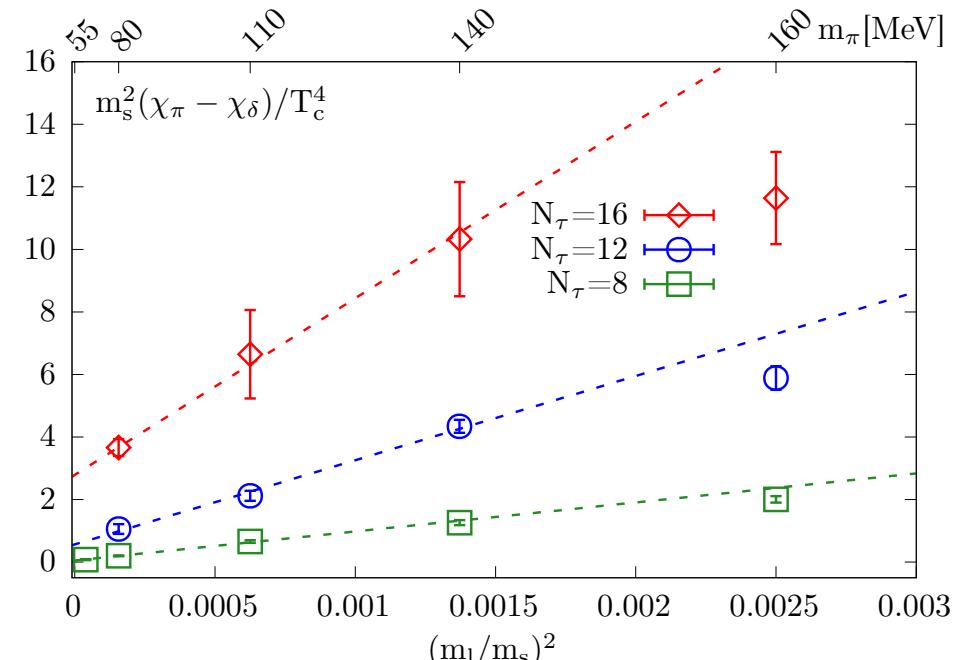
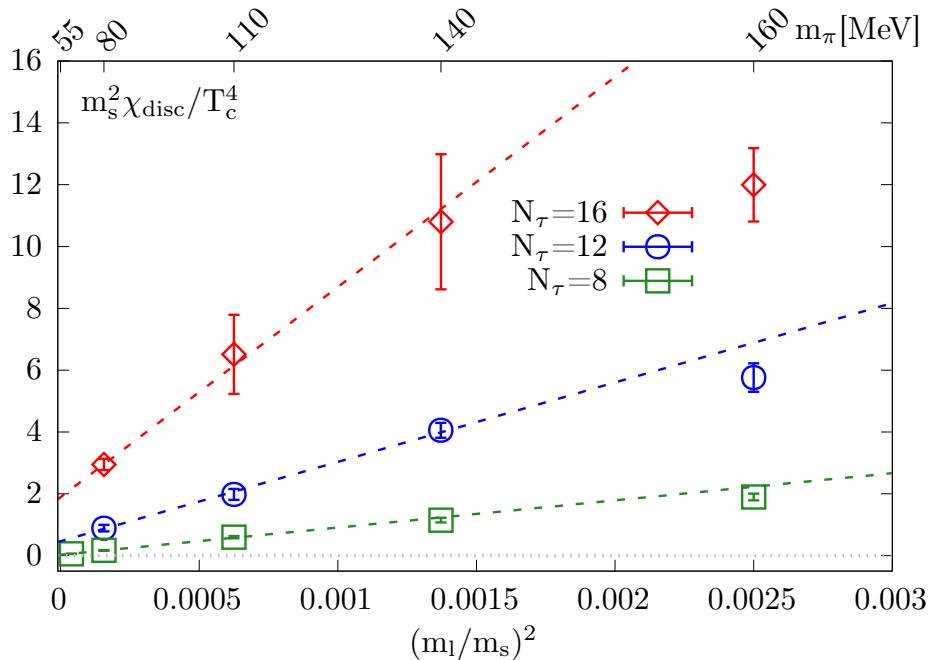
	χ^2/dof	
N_τ	Linear fits	Quadratic fits
8	0.43	13972.7
12	4.4	1504.0
16	0.1	198.5

Due to the restoration of $Z(2)$ subgroup of $SU_L(2) \times SU_R(2)$ symmetry, the partition function is an even function of m_l

$$\langle \bar{\psi} \psi \rangle \propto m_l \text{ as } m_l \rightarrow 0$$

$$\chi_{\text{disc}} \propto m_l^2 \text{ as } m_l \rightarrow 0$$

Two $U_A(1)$ measures



📌 Linear in m_l^2 at $m_\pi \lesssim 140$ MeV

Linear fits for $m_\pi \lesssim 140$ MeV data at each N_τ yield values at $m_l=0$:

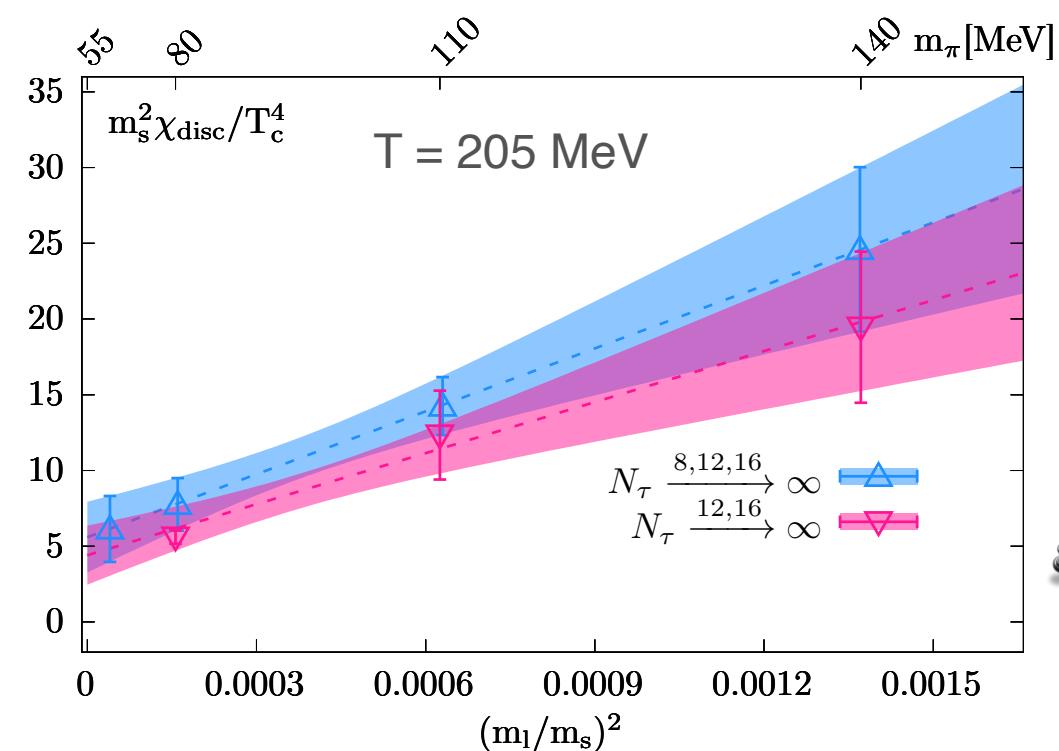
N_τ	$m_s^2 \chi_{disc}/T_c^4$	$m_s^2 (\chi_\pi - \chi_\delta)/T_c^4$
8	0.0030(7)	0.05(1)
12	0.47(8)	0.6(2)
16	1.9(1)	2.8(1)

In the chiral symmetric phase, $\chi_\pi - \chi_\delta$ should equal to χ_{disc} at $m_l=0$



Continuum extrapolations are crucial!

Continuum & chiral extrapolations of χ_{disc}



Joint fit: simultaneous fits

Continuum: $c_0 + c_1/N_\tau^2 + c_2/N_\tau^4$

Chiral: quadratic in quark mass

Value at $N_\tau \rightarrow \infty$ and $m_l \rightarrow 0$:

5.6 ± 2.3

Sequential fit: first continuum and then chiral extrapolations

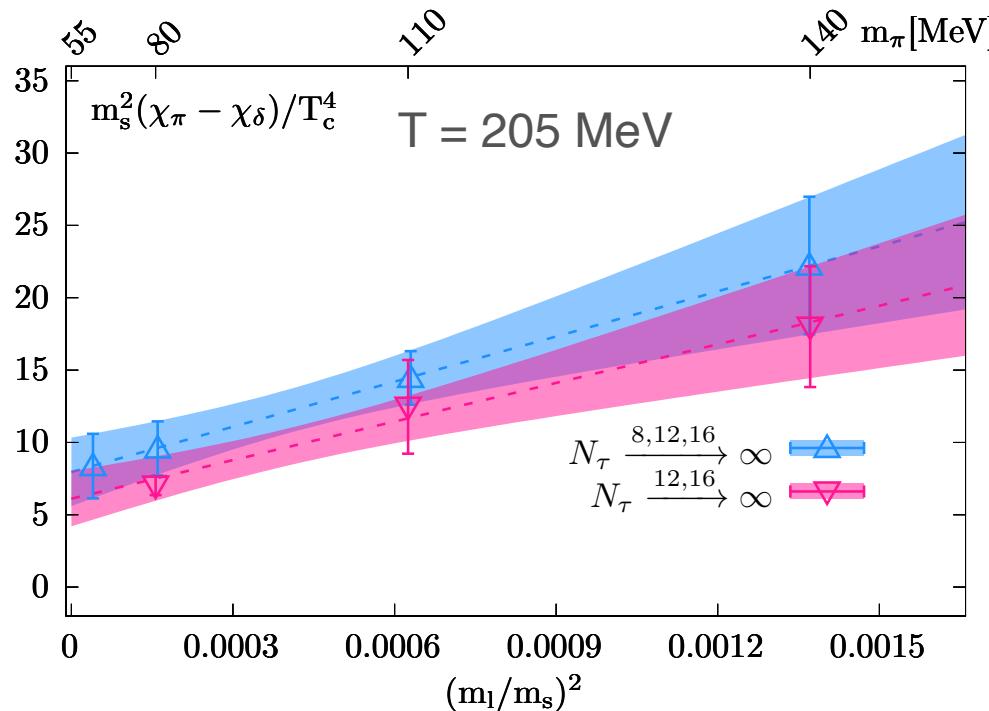
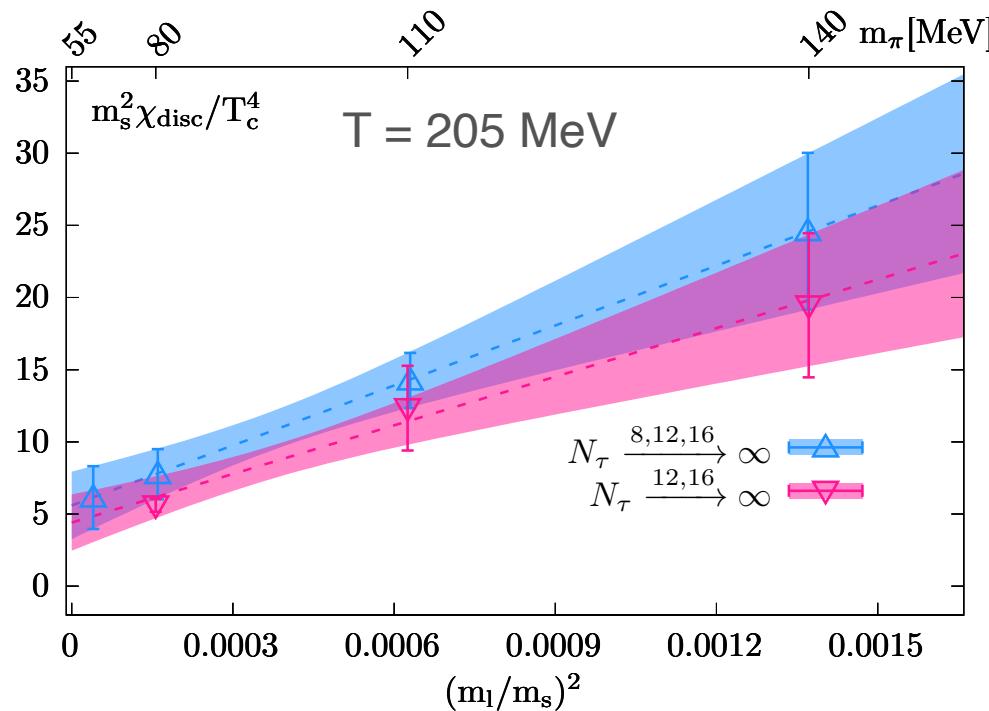
Continuum: quadratic in $1/N_\tau$ with $N_\tau=12&16$

Chiral: quadratic in quark mass

Value at $N_\tau \rightarrow \infty$ and $m_l \rightarrow 0$:

4.4 ± 1.9

Continuum & chiral extrapolations of two $U_A(1)$ measures



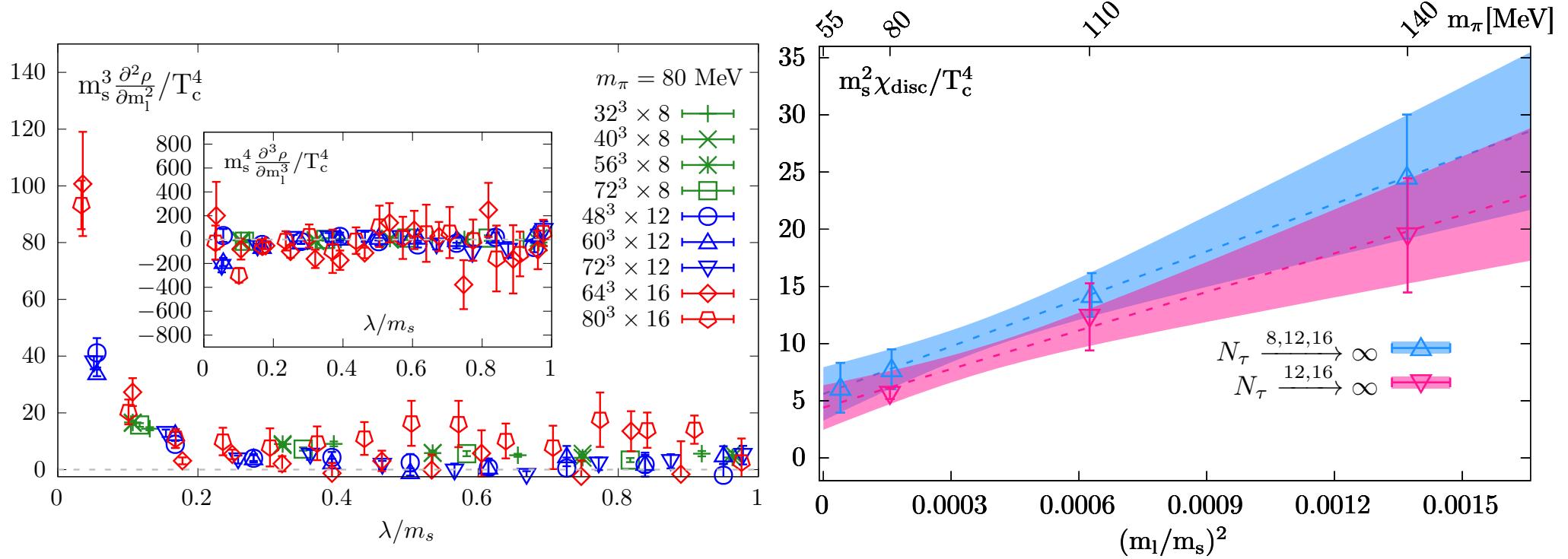
$N_\tau \rightarrow \infty$ and $m_l \rightarrow 0$	$m_s^2 \chi_{\text{disc}} / T_c^4$	$m_s^2 (\chi_\pi - \chi_\delta) / T_c^4$
Joint fit	5.6 ± 2.3	8.0 ± 2.4
Sequential fit	4.4 ± 1.9	6.1 ± 1.9

Axial anomaly remains manifested at $T \sim 1.6 T_c$
in the chiral limit at 2-3 sigma level

Summary & Conclusions

- We established novel relations between $\partial^n \rho / \partial^n m_l$ & C_{n+1}

In $N_f=2+1$ QCD at $T \sim 1.6T_c$



Summary & Conclusions

Our study suggests:

- ▶ At $T \sim 1.6 T_c$ the microscopic origin of axial anomaly is driven by the weakly interacting instanton gas motivated $\rho(\lambda \rightarrow 0, m_l \rightarrow 0) \propto m_l^2 \delta(\lambda)$
- ▶ $N_f=2+1$ QCD: 2nd order chiral phase transition belongs to 3-d O(4) universality class

Outlook:

- The methodology would be useful for other discretization schemes

Backup

Calculation of eigenvalue spectrum

- Commonly used method: Lanczos algorithm to calculate the individual low-lying eigenvalues
- Here we utilized the Chebyshev filtering technique combined with a stochastic estimate of the mode number

mode
number:

$$\bar{n}[s, t] \approx \frac{1}{N_r} \sum_{r=1}^{N_r} \sum_{j=0}^p g_j^p \gamma_j \langle \xi_r^\dagger T_j(A) \xi_r \rangle$$



Spectrum:

$$\rho(\lambda, \delta) = \frac{1}{V} \frac{\bar{n}[\lambda - \delta/2, \lambda + \delta/2]}{\delta} \quad (\lambda \geq \delta/2)$$

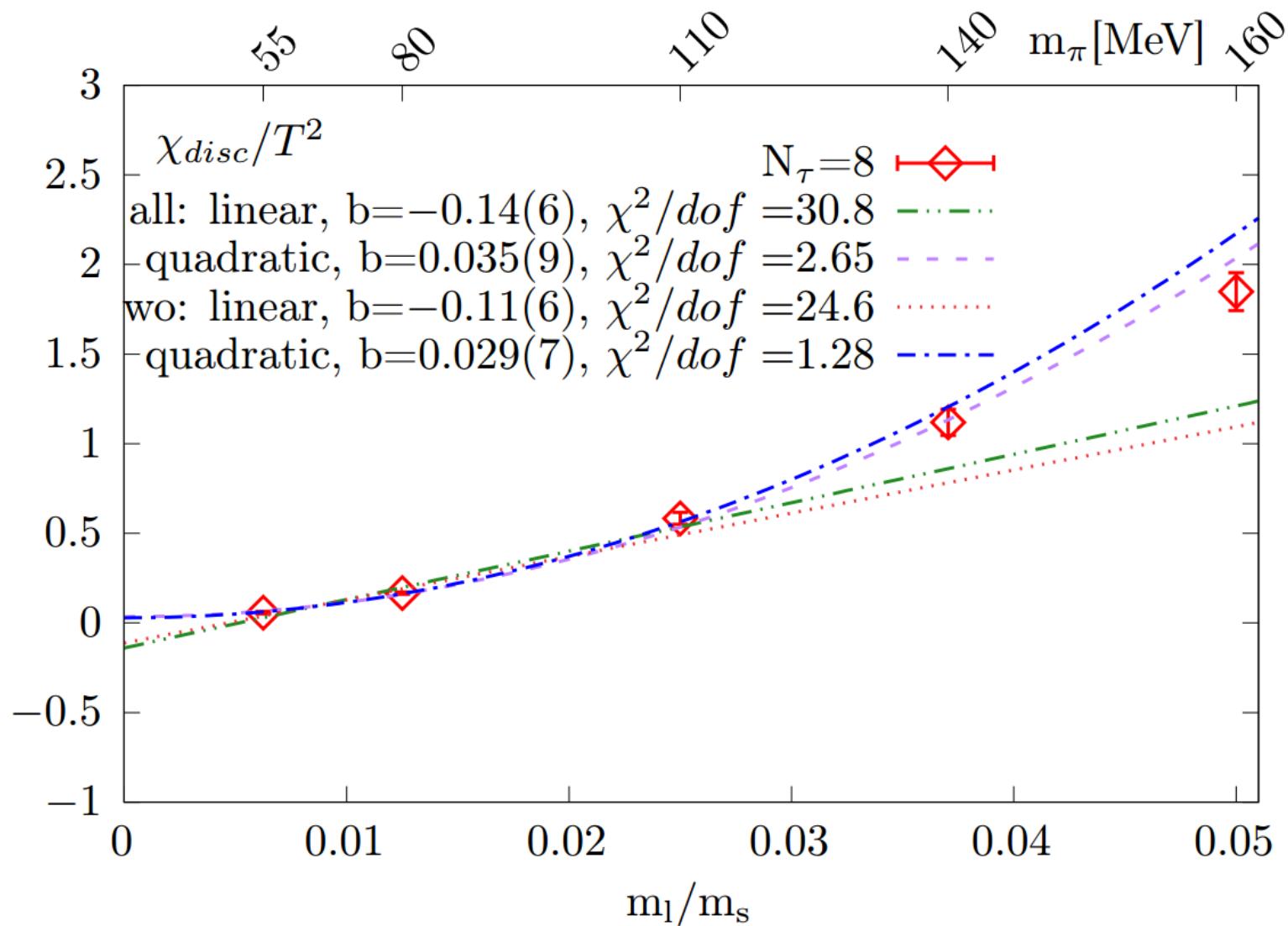
T_j : Chebyshev polynomial

γ_j : coefficient

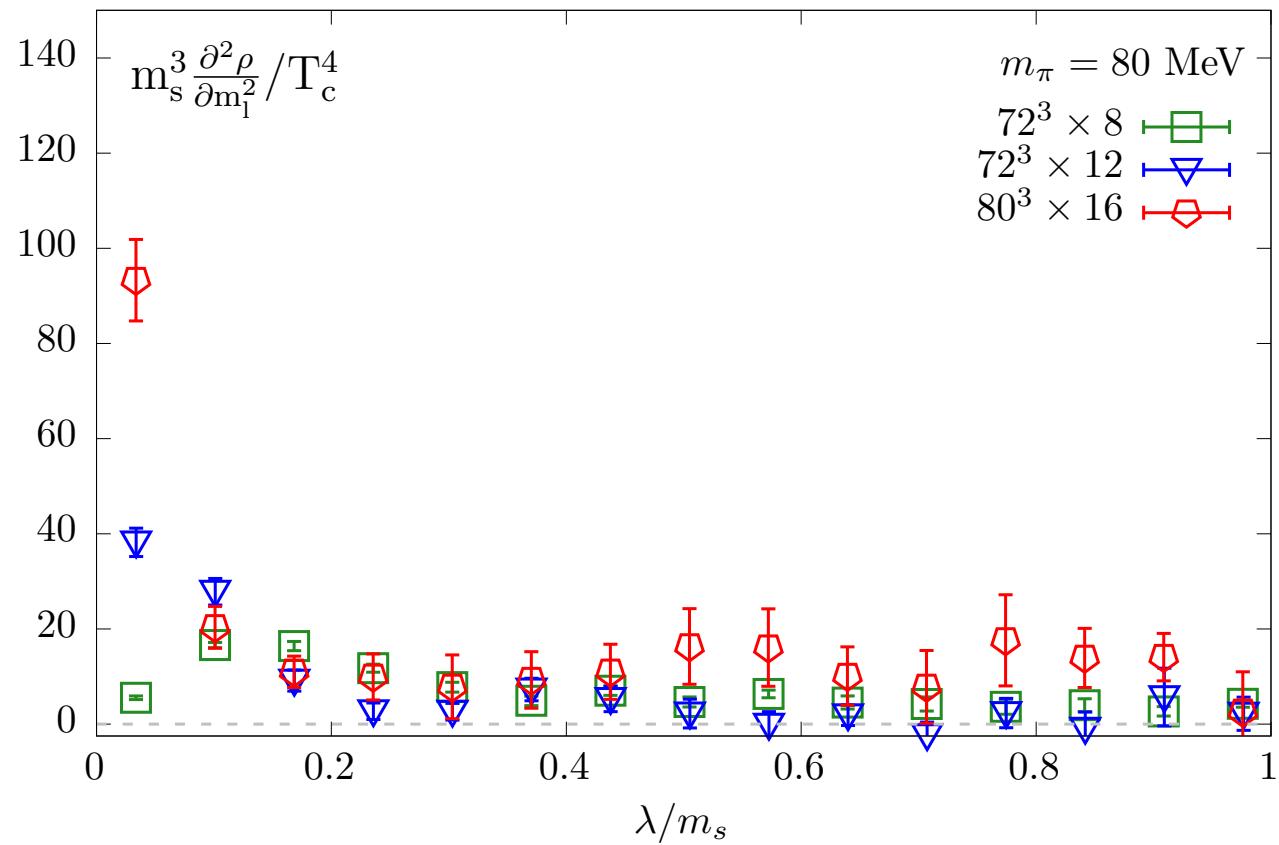
p : polynomial order

g_j^p : Jackson's dumping factor

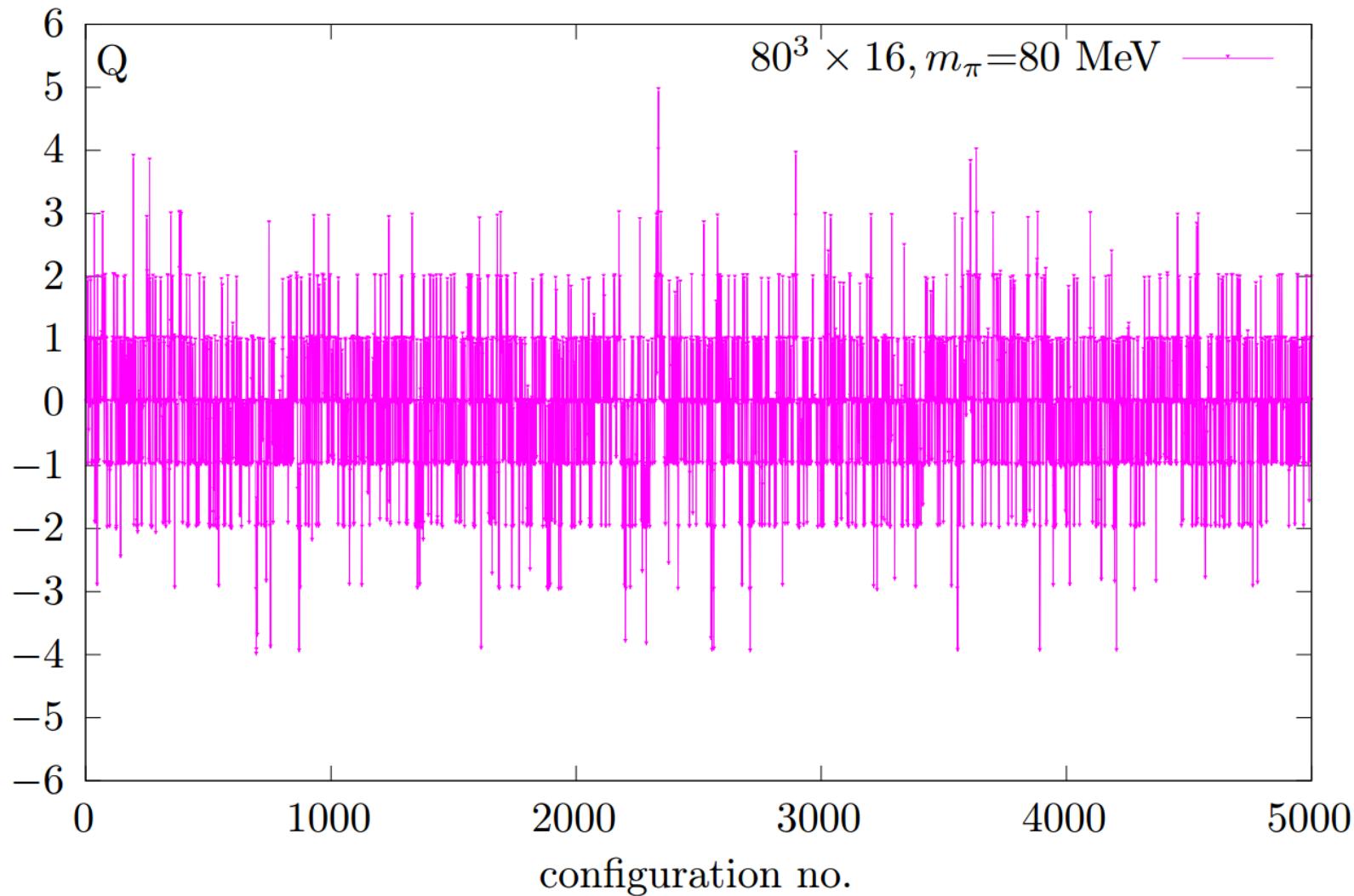
YuZhang, Lattice19', arXiv:2001.05217
Giusti and Luscher, arXiv:0812.3638
A.Patela, arXiv:1204.432
DiNapoli et al., arXiv: 1308.4275
Itou et al, arXiv:1411.1155
Fodor et al., arXiv:1605.08091
Cossu et al., arXiv:1601.00744



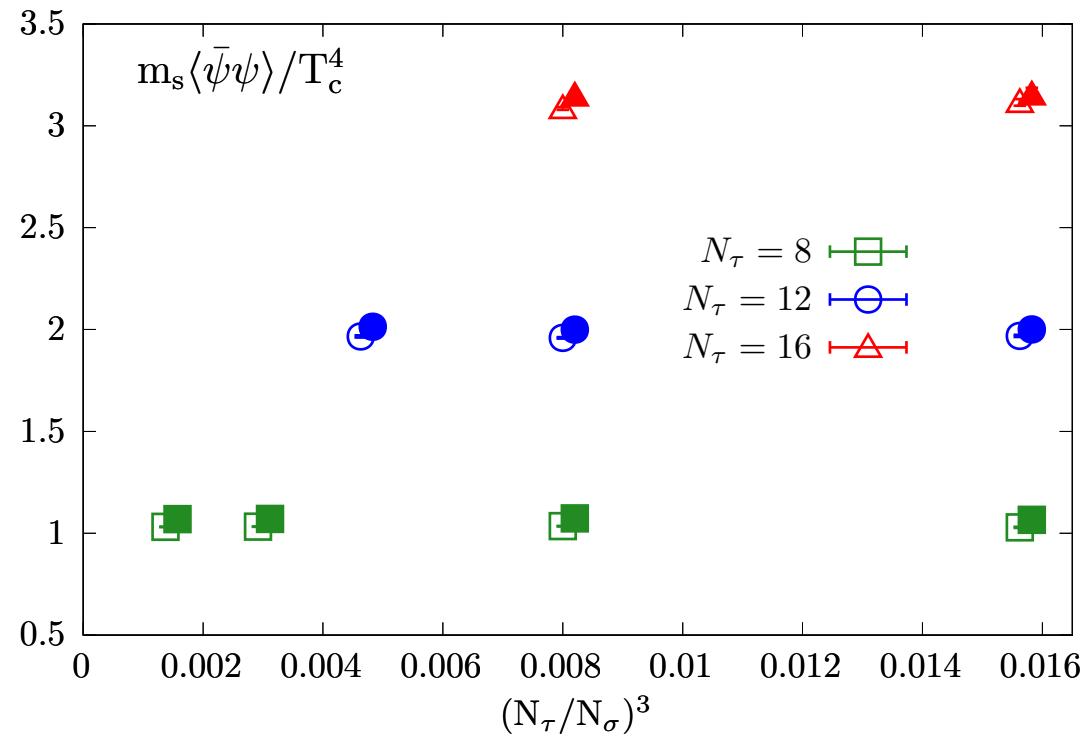
2nd mass derivative of ρ



Time history of the topological charge



Chiral condensate

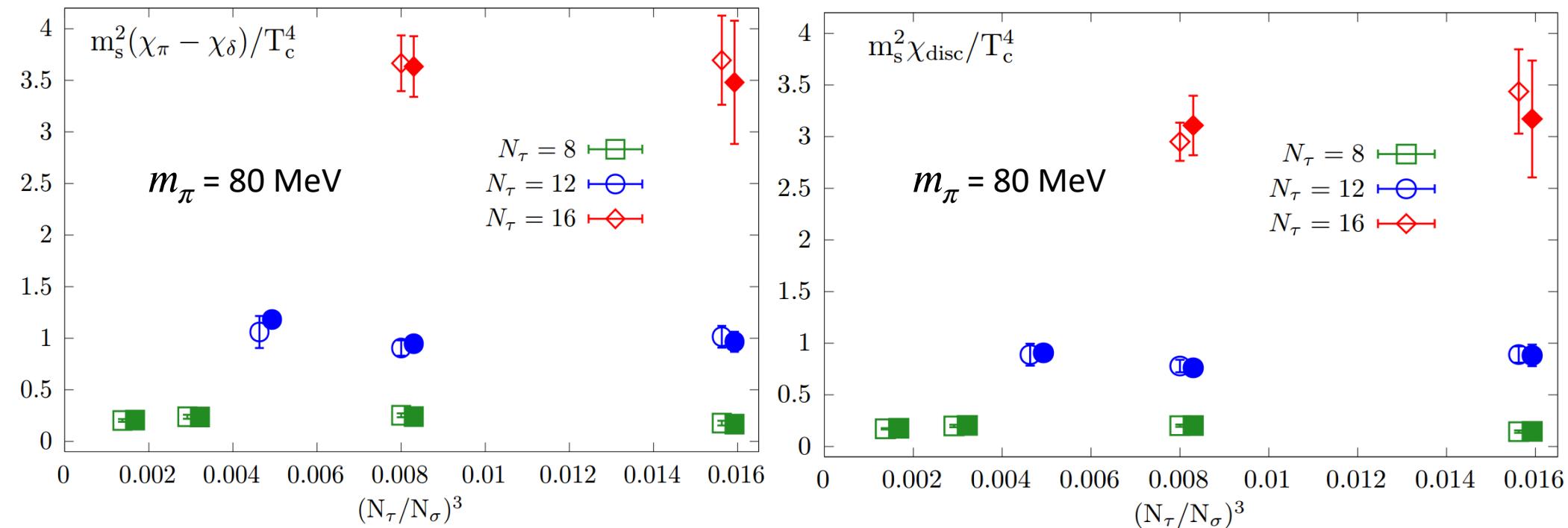


$$\langle \bar{\psi} \psi \rangle = \int_0^\infty d\lambda \frac{4m_l \rho}{\lambda^2 + m_l^2} + \frac{2T}{V} \frac{\langle |Q_{\text{top}}| \rangle}{m}$$

$$\langle |Q_{\text{top}}| \rangle \propto \sqrt{V}$$

Zero mode contribution vanishes in the thermodynamical limit

Volume dependence of two $U_A(1)$ measures



Volume dependences is very small

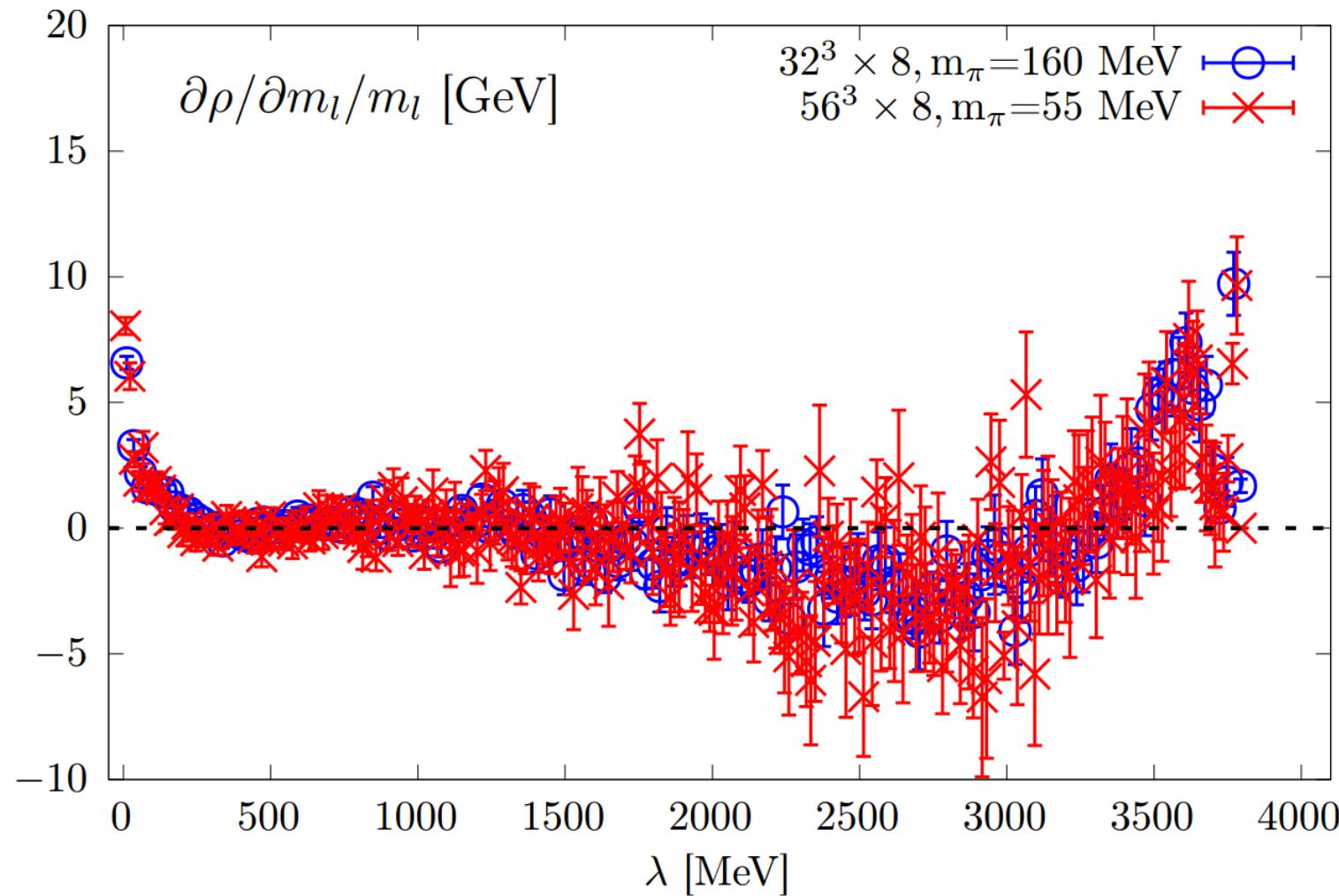
Poisson distribution

$$\begin{aligned}
C_2(\lambda, \lambda') &= \langle \rho_u(\lambda) \rho_u(\lambda') \rangle - \langle \rho_u(\lambda) \rangle \langle \rho_u(\lambda') \rangle \\
&= (\frac{1}{V})^2 \left\langle \sum_{k=1}^N \delta(\lambda - \lambda_k) \sum_{l=1}^N \delta(\lambda' - \lambda_l) \right\rangle - \langle \rho_u(\lambda) \rangle \langle \rho_u(\lambda') \rangle \\
&= (\frac{1}{V})^2 \left\langle \sum_{k=1}^N \delta(\lambda - \lambda_k) \delta(\lambda' - \lambda_k) \right\rangle + (\frac{1}{V})^2 \left\langle \sum_{k=1}^N \delta(\lambda - \lambda_k) \sum_{l \neq k} \delta(\lambda' - \lambda_l) \right\rangle - \langle \rho_u(\lambda) \rangle \langle \rho_u(\lambda') \rangle^{(1)} \\
&= \frac{1}{V} \langle \rho_u(\lambda) \rangle \delta(\lambda - \lambda') + (\frac{1}{V})^2 \left\langle \sum_{k=1}^N \delta(\lambda - \lambda_k) \right\rangle \left\langle \sum_{l \neq k} \delta(\lambda' - \lambda_l) \right\rangle - \langle \rho_u(\lambda) \rangle \langle \rho_u(\lambda') \rangle
\end{aligned}$$

$$\frac{1}{V} \left\langle \sum_{l \neq k} \delta(\lambda' - \lambda_l) \right\rangle = \frac{N-1}{N} \langle \rho_u(\lambda') \rangle \quad (N = V/2)$$

$$C_2(\lambda, \lambda') = \frac{1}{V} \langle \rho_u(\lambda) \rangle \delta(\lambda - \lambda') - \frac{1}{N} \langle \rho_u(\lambda) \rangle \langle \rho_u(\lambda') \rangle$$

Quark mass dependence of $m_l^{-1}(\partial\rho/\partial m_l)$



Quark mass dependence of $\partial^2\rho/\partial m_l^2$

