Renormalon subtraction using Fourier transform
— towards precise QCD calculation

Yuuki Hayashi (Tohoku University D2)

Based on PLB 819 (2021) 136414 YH, Yukinari Sumino, Hiromasa Takaura
arXiv: 2106.03687 YH, Yukinari Sumino, Hiromasa Takaura

Table of contents

1. Introduction: Ambiguity in QCD — renormalon problem

- Conventional framework of QCD calculation and its problem
- Modern approach to precise QCD calculation
- Motivation of our research

2. Method: Renormalon subtraction using Fourier transform

3. Application: B, D meson mass

4. Conclusions

【Introduction】

- **Recent progress of perturbative QCD computation**

Computation technique, High power machine, Clever algorithm,...

→ *Higher order perturbative calculation*

- Beta function $O(\alpha_s^6)$ 2016 Baikov, *et al*, 2017 Herzog, *et al*, 2017 Luthe, *et al*
- Adler function (Cross section of $e^+e^- \to \text{hadrons}$) $O(\alpha_s^4)$ 2012 Baikov, Chetyrkin, Kuhn
- static QCD potential $O(\alpha_s^4)$ 2009 Anzai, Kiyo, Sumino, 2009 Smirnov, Smirnov, Steinhauser
- Pole - $\overline{\text{MS}}$ quark mass relation $O(\alpha_s^4)$ 2016 Marquard, Smirnov, Smirnov, Steinhauser, Wellmann

【Introduction】

- **Operator product expansion (OPE)**

Incorporate the Non-PT corrections in $1/Q$ expansion based on low energy EFT

$$[X(Q)]_{\mathrm{OPE}} = \boxed{C_1(Q)} + \boxed{C_{\mathcal{O}_d}(Q)} \boxed{\frac{\langle \mathcal{O}_d \rangle}{Q^d}} + \cdots$$

$X(Q)$: dimensionless observable

Q : typical hard scale of X

Perturbative(PT) + Non-perturbative(Non-PT)

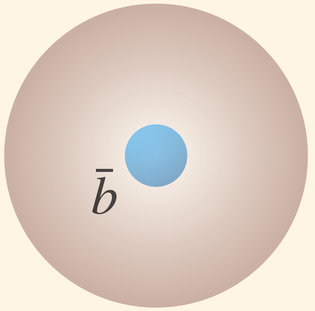
$\boxed{C_1(Q),\ C_{\mathcal{O}_d}(Q)}$: Wilson coefficient (perturbative correction)

$\boxed{\mathcal{O}_d}$: d -dim operator in low energy EFT $\langle \mathcal{O}_d \rangle = O\left(\Lambda_{\mathrm{QCD}}^d\right)$

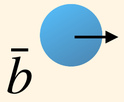
(Λ_{QCD} : non-perturbative scale of QCD)

【Introduction】

- **Example of OPE: B meson mass**



Light degrees of freedom



Non-relativistic bottom quark

$$M_B = \langle B | \hat{H}_{\rm QCD} | B \rangle \xrightarrow{m_b \to \infty} M_B = \langle B | \hat{H}_{\rm light} | B \rangle + \langle B | \hat{H}_{\rm HQET} | B \rangle$$

$$= O(\Lambda_{\rm QCD})$$

B: at rest

heavy b quark + light sector

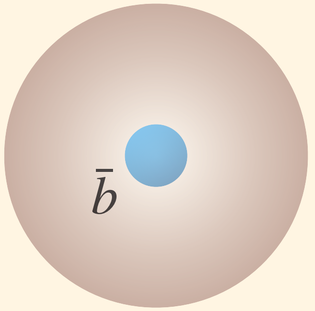
Heavy quark effective theory (HQET)

$$\hat{H}_{\rm HQET} = m_b b^\dagger b + \frac{1}{2m_b}\left(b^\dagger \left(-\vec{D}^2\right) b + C_{cm}\, b^\dagger g_s \sigma \cdot B\, b \right) + O\left(1/m_b^2\right)$$

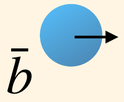
C_{cm}: obtained from matching to full QCD at $\mu = m_b$

【Introduction】

- **Example of OPE: B meson mass**



Light degrees of freedom



Non-relativistic bottom quark

$$\langle B|\hat{H}_{\rm light}|B\rangle = \overline{\Lambda}$$

Light degrees of freedom

$$\langle B|\hat{H}_{\rm HQET}|B\rangle = m_b + \frac{\mu_\pi^2}{2m_b} + C_{cm}\frac{\mu_G^2}{2m_b} + \ldots$$

$$\langle B|b^\dagger b|B\rangle = 1$$

$$\langle B|b^\dagger(-\vec{D}^2)b|B\rangle = \mu_\pi^2$$

Kinetic energy of b quark

$$\langle B|b^\dagger g_s \sigma\cdot B\, b|B\rangle = \mu_G^2$$

Chromo-magnetic interaction

$$\Rightarrow [M_B]_{\rm OPE} = m_b + \overline{\Lambda} + \frac{\mu_\pi^2}{2m_b} + C_{cm}\frac{\mu_G^2}{2m_b} + O(1/m_b^2)$$

【Introduction】

- **Spin dependence**

OPE formula for $B^{(*)}$ meson

$$\left[M_{B^{(*)}}\right]_{\mathrm{OPE}} = m_b + \overline{\Lambda} + \frac{\mu_\pi^2}{2m_b} + A(J)C_{cm}(m_b)\frac{\mu_G^2}{2m_b} + \ldots$$

Spin average

$$A(J) = \begin{cases} -1/3 & (J=1) \text{ Vector meson } B^* \\ 1 & (J=0) \text{ Pseudoscalar meson } B \end{cases}$$

$$\left[\overline{M}_B\right]_{\mathrm{OPE}} = \left[\frac{M_B + 3M_{B^*}}{4}\right]_{\mathrm{OPE}} = m_b + \overline{\Lambda} + \frac{\mu_\pi^2}{2m_b} + O\left(1/m_b^2\right)$$

【Introduction】

- **Heavy-quark symmetry**

Considering HQET with 2 heavy quark (b, c quarks)

OPE formula for B meson

$$[\overline{M}_B]_{\mathrm{OPE}} = m_b + \boxed{\overline{\Lambda}} + \boxed{\frac{\mu_\pi^2}{2m_b}} + \ldots$$

Same formula for D meson (same quantum number)

$$[\overline{M}_D]_{\mathrm{OPE}} = m_c + \boxed{\overline{\Lambda}} + \boxed{\frac{\mu_\pi^2}{2m_c}} + \ldots$$

$\boxed{}$: Universal non-PT parameter → describe various observables

(Light sector is insensitive to heavy quark.)

【Introduction】

- **Fate of QCD perturbation**

$$[X(Q)]_{\rm OPE} = \boxed{C_1(Q)} + \boxed{C_{\mathcal{O}_d}(Q)} \frac{\boxed{\langle \mathcal{O}_d \rangle}}{Q^d} + \cdots$$

Wilson coefficient (UV contribution): Perturbative QCD calculation

$$\boxed{C_1(Q)} \underset{\rm PT}{=} c_0 \alpha_s + c_1 \alpha_s^2 + \cdots$$

・Asymptotic freedom $\alpha_s(\mu) \to 0$ as $\mu \to \infty$

→ Perturbation can systematically approximate the "true" QCD correction ?

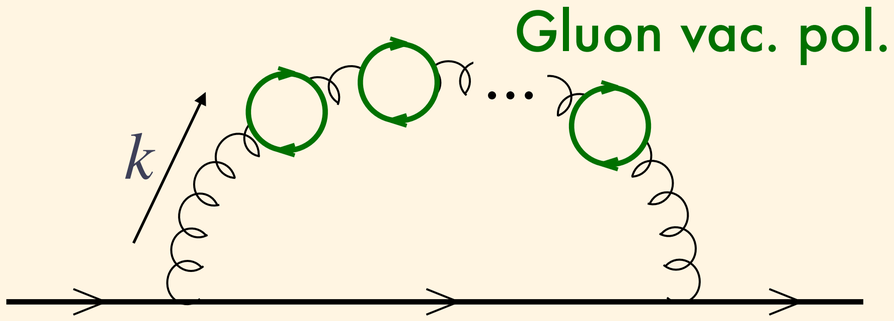
NO

【Introduction】

- **Fate of QCD perturbation → divergent series**

$$b_0 = \frac{1}{4\pi}\left(11 - \frac{2}{3}n_l\right)$$

e.g.



$$\sim \int_0^\infty dk_E^2\, f(k_E^2)\left[-\log(k_E^2)\right]^n b_0^n \alpha_s^{n+1}$$

$$f(k_E^2) \sim (k_E^2)^{u_0-1} \quad \text{as} \quad k_E \to 0$$

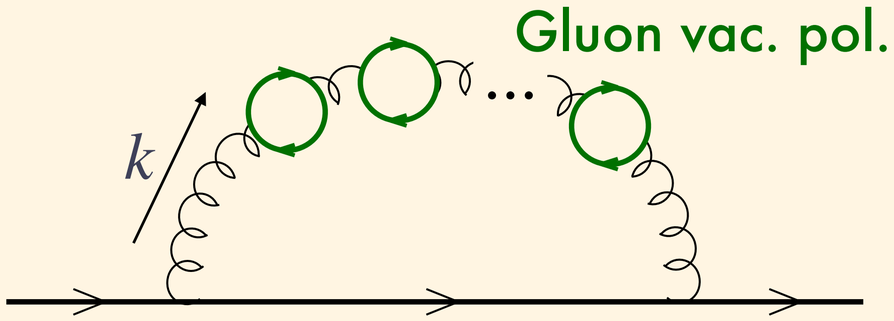
$$(u_0 > 0)$$

【Introduction】

- **Fate of QCD perturbation → divergent series**

$$b_0 = \frac{1}{4\pi}\left(11 - \frac{2}{3}n_l\right)$$

e.g.



$$\sim \int_0^\infty dk_E^2\, f(k_E^2)\left[-\log(k_E^2)\right]^n b_0^n \alpha_s^{n+1}$$

$$f(k_E^2) \sim (k_E^2)^{u_0-1} \quad \text{as} \quad k_E \to 0$$

$$(u_0 > 0)$$

Infrared (IR) region $\sim \int_{\mathrm{IR}} dk_E^2\, (k_E^2)^{u_0-1}\left[-\log(k_E^2)\right]^n b_0^n \alpha_s^{n+1} \sim \left(\frac{b_0}{u_0}\right)^n n!\,\alpha_s^{n+1}$

Renormalon problem

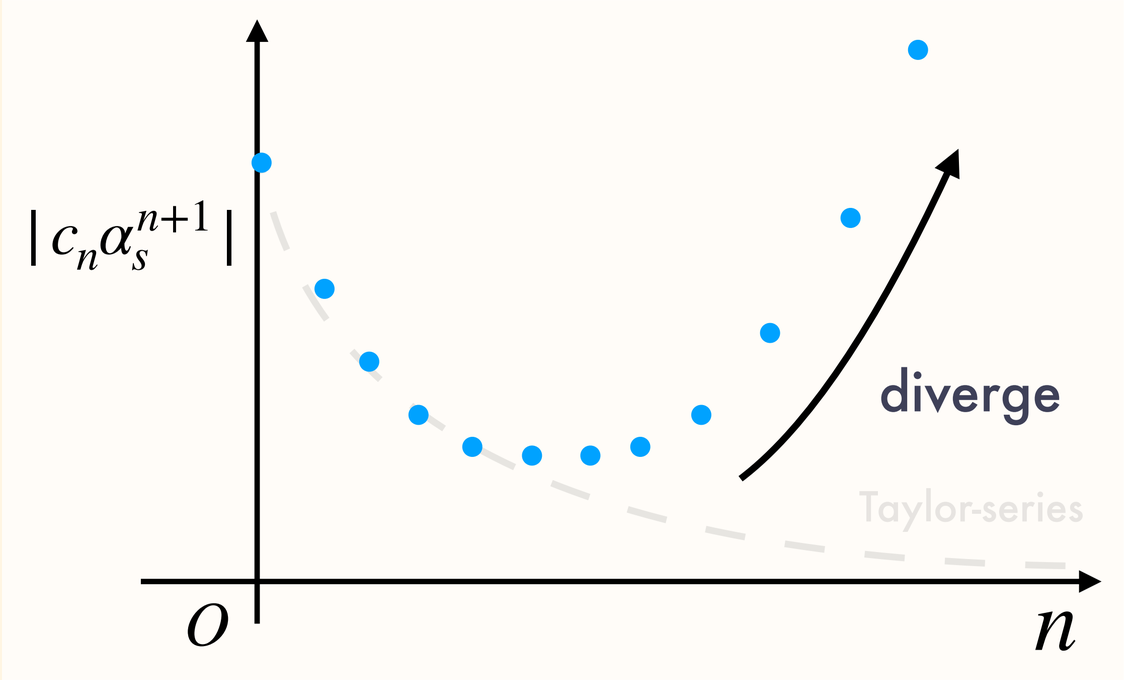
【Introduction】

- **Ambiguity induced from renormalons**

u_0 : constant related to IR

$$C_1(Q) \underset{\text{PT}}{=} \sum_{n=0}^{\infty} c_n \alpha_s^{n+1} \xrightarrow[(n \gg 1)]{\text{Asymptotic behavior}}$$

$$\boxed{c_n \sim n!(b_0/u_0)^n}$$



【Introduction】

- **Ambiguity induced from renormalons**

u_0 : constant related to IR

$$C_1(Q) \underset{\text{PT}}{=} \sum_{n=0}^{\infty} c_n \alpha_s^{n+1} \xrightarrow[(n \gg 1)]{\text{Asymptotic behavior}}$$

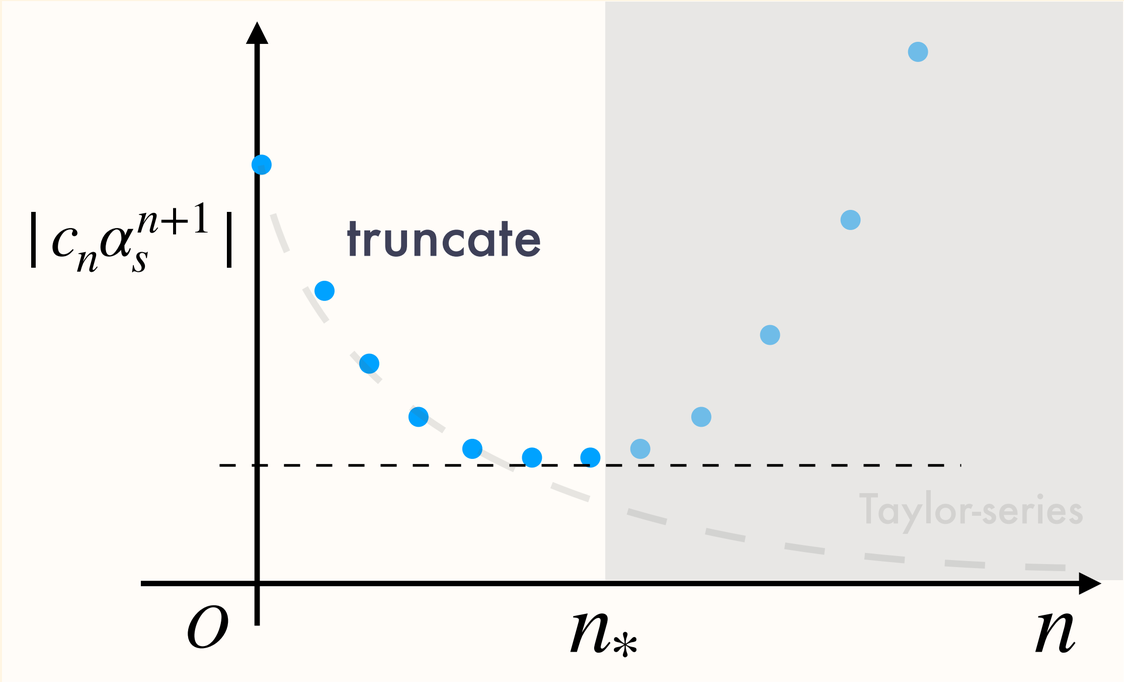
$$\boxed{c_n \sim n!(b_0/u_0)^n}$$

↓ Truncate series at $n = n_*$ (minimal value)

$$|c_{n_*-1}\alpha_s^{(n_*-1)+1}| \approx |c_{n_*}\alpha_s^{n_*+1}|$$

$$\therefore n_* \approx \frac{u_0}{b_0 \alpha_s}$$

$$|c_n \alpha_s^{n+1}| \geq |c_{n_*}\alpha_s^{n_*+1}| \approx \left(\Lambda_{\text{QCD}}/Q\right)^{2u_0}$$



【Introduction】

- **Ambiguity induced from renormalons**

u_0 : constant related to IR

$$C_1(Q) \underset{\text{PT}}{=} \sum_{n=0}^{\infty} c_n \alpha_s^{n+1} \xrightarrow[(n \gg 1)]{\text{Asymptotic behavior}}$$

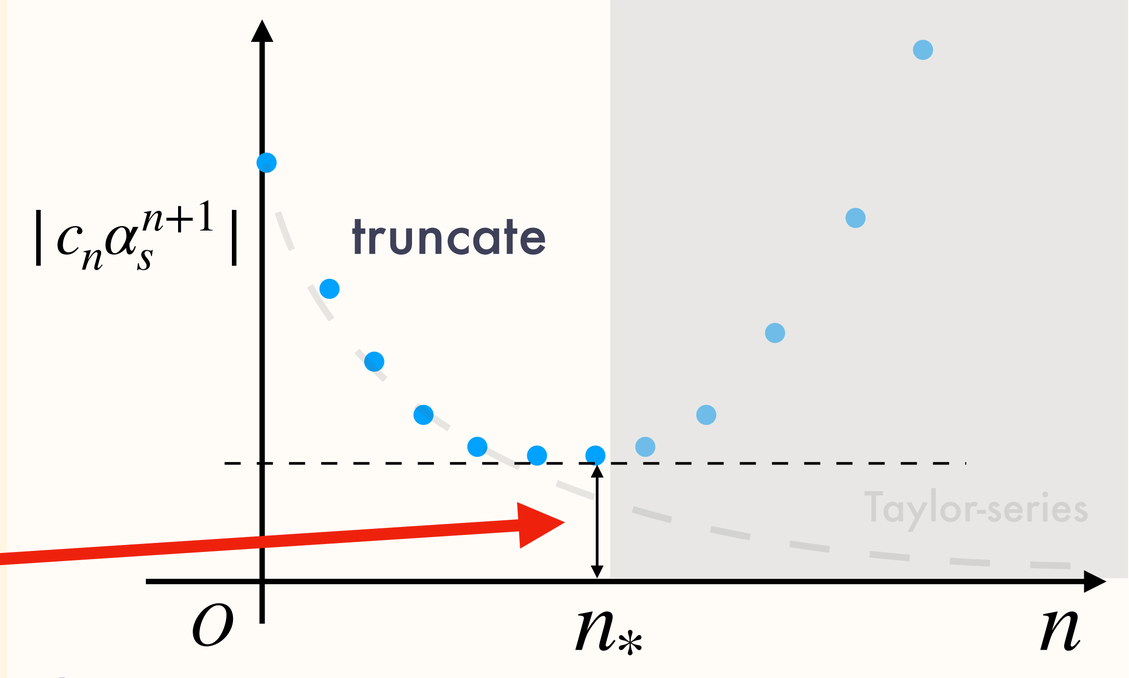
$$c_n \sim n!(b_0/u_0)^n$$

↓ Truncate series at $n = n_*$ (minimal value)

$$|c_{n_*-1}\alpha_s^{(n_*-1)+1}| \approx |c_{n_*}\alpha_s^{n_*+1}|$$

$$\therefore n_* \approx \frac{u_0}{b_0 \alpha_s}$$

$$|c_n \alpha_s^{n+1}| \geq |c_{n_*}\alpha_s^{n_*+1}| \approx \boxed{(\Lambda_{\text{QCD}}/Q)^{2u_0}}$$



δC_1 : ambiguity induced from renormalon

【Introduction】

- **Renormalon problem in OPE**

Experiment ⟷ $$[X(Q)]_{\mathrm{OPE}} = \boxed{C_1(Q)} + \boxed{C_{\mathcal{O}_d}(Q)} \boxed{\frac{\langle \mathcal{O}_d \rangle}{Q^d}} + \cdots$$

Ambiguous **Ambiguous**

$$\delta C_1 \approx \left(\Lambda_{\mathrm{QCD}}/Q\right)^{2u_0}$$

【Introduction】

- **Renormalon problem in OPE**

Ambiguous

Experiment ⟷ $$[X(Q)]_{\mathrm{OPE}} = \boxed{C_1(Q)} + \boxed{C_{\mathcal{O}_d}(Q)} \boxed{\frac{\langle \mathcal{O}_d \rangle}{Q^d}} + \cdots$$

Ambiguous Ambiguous

$$\delta C_1 \approx \left(\Lambda_{\mathrm{QCD}}/Q\right)^{2u_0}$$

→ Non-PT parameter $\langle \mathcal{O}_d \rangle$ is also ambiguous.

(Ambiguity from $u_0 = d/2$ renormalon mix with $\langle \mathcal{O}_d \rangle$)

【Introduction】

- **Modern way to subtract renormalon in OPE framework**

Renormalon ambiguity comes from IR structure of X → related to low energy physics

$$[X(Q)]_{\mathrm{OPE}} = \boxed{C_1(Q)} + \boxed{C_{\mathcal{O}_d}(Q)} \boxed{\frac{\langle \mathcal{O}_d \rangle}{Q^d}} + \cdots$$

【Introduction】

- **Modern way to subtract renormalon in OPE framework**

Renormalon ambiguity comes from IR structure of X → related to low energy physics

$$[X(Q)]_{\mathrm{OPE}} = C_1(Q) + C_{\mathcal{O}_d}(Q)\frac{\langle \mathcal{O}_d \rangle}{Q^d} + \cdots$$

①Separate ambiguous part

$$[C_1]_{\mathrm{RS}} + \delta C_1$$

$$[C_{\mathcal{O}_d}]_{\mathrm{RS}} + \delta C_{\mathcal{O}_d}$$

RS: renormalon subtracted

【Introduction】

- **Modern way to subtract renormalon in OPE framework**

Renormalon ambiguity comes from IR structure of X → related to low energy physics

$$[X(Q)]_{\mathrm{OPE}} = C_1(Q) + C_{\mathcal{O}_d}(Q)\frac{\langle \mathcal{O}_d \rangle}{Q^d} + \ldots$$

②Cancel ambiguous part

w/ non-PT matrix elements

1985 Mueller

$$[C_1]_{\mathrm{RS}} + \delta C_1$$

$$[C_{\mathcal{O}_d}]_{\mathrm{RS}} + \delta C_{\mathcal{O}_d}$$

RS: renormalon subtracted

【Introduction】

- **Modern way to subtract renormalon in OPE framework**

Renormalon ambiguity comes from IR structure of X → related to low energy physics

③Determine the renormalon-subtracted OPE

$$[X(Q)]_{\mathrm{OPE}} = [C_1(Q)]_{\mathrm{RS}} + [C_{\mathcal{O}_d}(Q)]_{\mathrm{RS}} \frac{\langle \mathcal{O}_d \rangle_{\mathrm{RS}}}{Q^d} + \cdots$$

RS: renormalon subtracted

$[\quad]_{\mathrm{RS}}$ Convergent series

$\langle \quad \rangle_{\mathrm{RS}}$ Improved non-PT matrix element

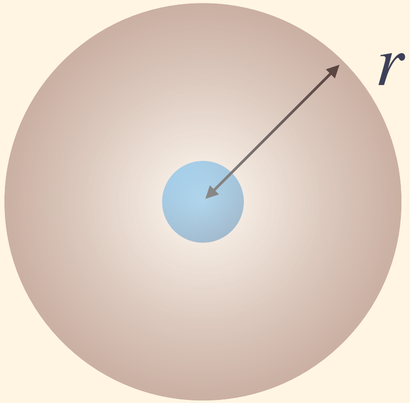
} Precise QCD calc.

【Introduction】

- **Back to B meson mass**

$$[\overline{M}_B]_{\mathrm{OPE}} = \left[\frac{M_B + 3M_{B^*}}{4}\right]_{\mathrm{OPE}} = m_b + \overline{\Lambda} + \frac{\mu_\pi^2}{2m_b} + O(1/m_b^2)$$

m_b : pole mass of bottom quark (on-shell) $\approx$ constituent quark mass



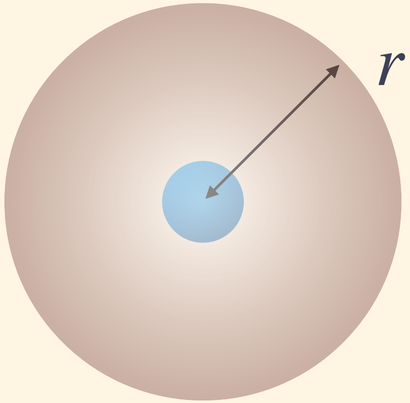
constituent quark

【Introduction】

- **Back to B meson mass**

$$[\overline{M}_B]_{\rm OPE} = \left[\frac{M_B + 3M_{B^*}}{4}\right]_{\rm OPE} = m_b + \overline{\Lambda} + \frac{\mu_\pi^2}{2m_b} + O(1/m_b^2)$$

m_b : pole mass of bottom quark (on-shell) $\approx$ constituent quark mass



$r \sim 1/\Lambda_{\rm QCD}$

constituent quark

→ Conceptually ambiguous due to light degrees of freedom

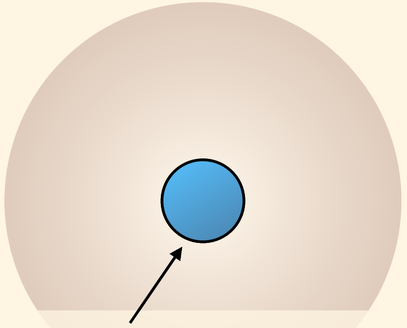
$$\delta m_b \sim O(\Lambda_{\rm QCD})$$

【Introduction】

- **Renormalons of quark pole mass**

$$[\overline{M}_B]_{\rm OPE} = \left[\frac{M_B + 3M_{B^*}}{4}\right]_{\rm OPE} = \boxed{m_b} + \boxed{\overline{\Lambda}} + \boxed{\frac{\mu_\pi^2}{2m_b}} + O\left(1/m_b^2\right)$$

m_b : pole mass of bottom quark (on-shell) $\approx$ constituent quark mass



current quark $\approx \bar{m}_b$

【pole-$\overline{\rm MS}$ relation】

$$m_b/\bar{m}_b \approx 1 + 0.424\alpha_s + 1.037\alpha_s^2 + 3.744\alpha_s^3 + 17.44\alpha_s^4 + \underset{\text{(est)}}{97.5872\alpha_s^5} + \cdots$$

Ambiguities $O(\Lambda_{\rm QCD}),\ O(\Lambda_{\rm QCD}^2/m_b),\ \cdots$

【Introduction】

- **Renormalons of quark pole mass**

$$[\overline{M}_B]_{\rm OPE} = \left[\frac{M_B + 3M_{B^*}}{4}\right]_{\rm OPE} = m_b + \overline{\Lambda} + \frac{\mu_\pi^2}{2m_b} + O(1/m_b^2)$$

Cancel with $\boxed{}$ → well defined

→ Precise prediction of various observables

e.g. D meson mass, $\Gamma(B \to X_c \ell \nu)$

nstituen quark mass

$3.744\alpha_s^3 - 17.44\alpha_s^4 + 97.5872\alpha_s^5 + \cdots$ (est)

Ambiguities $O(\Lambda_{\rm QCD}),\ O(\Lambda_{\rm QCD}^2/m_b),\ \cdots$

current quark $\approx \bar{m}_b$

【Introduction】

- **Short summary and motivation**

QCD effect $\xrightarrow{\text{OPE}}$ Perturbative (**Ambiguous**) + Non-perturbative (**Ambiguous**)

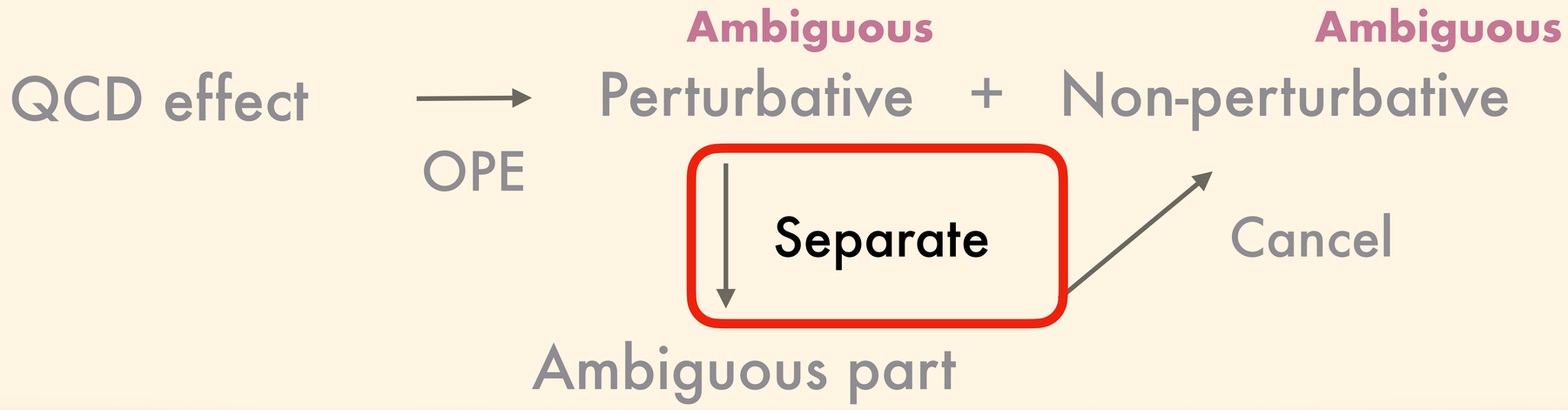
Perturbative $\xrightarrow{\text{Separate}}$ Ambiguous part

Ambiguous part $\xrightarrow{\text{Cancel}}$ Non-perturbative

$\longrightarrow$ $[\text{Perturbative}]_{\text{RS}}$ (**Convergent**) + $[\text{Non-perturbative}]_{\text{RS}}$ (**Improved**)

【Introduction】

- **Short summary and motivation**



【Motivation: How to separate renormalons?】

Conceptual procedure: not practical, needs approximation

→ Our new method: improve approx accuracy systematically.

Table of contents

1. Introduction: Ambiguity in QCD — renormalon problem

How to separate renormalons from Wilson coefficient ?

2. Method: Renormalon subtraction using Fourier transform

- Conceptual procedure and its problem
- Property of our method and toy model demonstration

3. Application: B, D meson mass

4. Conclusions

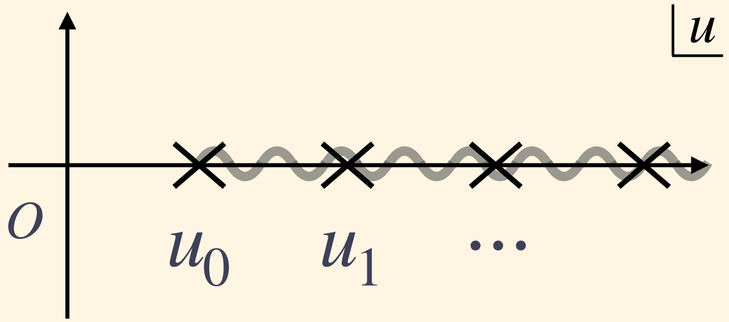
【Method】

- **Conceptual procedure for renormalon subtraction**

Borel transform of $C_1(Q) \underset{\text{PT}}{=} \sum_{n=0}^{\infty} c_n \alpha_s^{n+1}$

$$\Rightarrow B_{C_1}(u) = \sum_{n=0}^{\infty} \frac{c_n}{n!}(u/b_0)^n$$

$$\left(\sum_{n=0}^{\infty} \frac{n!(b_0/u_0)^n}{n!}(u/b_0)^n = \frac{1}{1-u/u_0} \right)$$



pole singularities & discontinuity

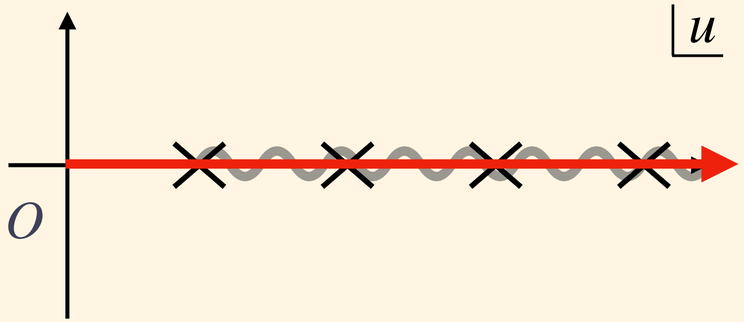
【Method】

- **Conceptual procedure for renormalon subtraction**

Borel transform of $C_1(Q) \underset{\text{PT}}{=} \sum_{n=0}^{\infty} c_n \alpha_s^{n+1}$

$$\Rightarrow B_{C_1}(u) = \sum_{n=0}^{\infty} \frac{c_n}{n!}(u/b_0)^n$$

$$\left(\sum_{n=0}^{\infty} \frac{n!(b_0/u_0)^n}{n!}(u/b_0)^n = \frac{1}{1-u/u_0} \right)$$



pole singularities & discontinuity

$$C_1(Q) \text{ “=” } \frac{1}{b_0}\int_0^{\infty} du\, e^{-\frac{u}{b_0\alpha_s}} B_{C_1}(u)$$

: Divergent due to singularity

(Reflection of renormalon divergence)

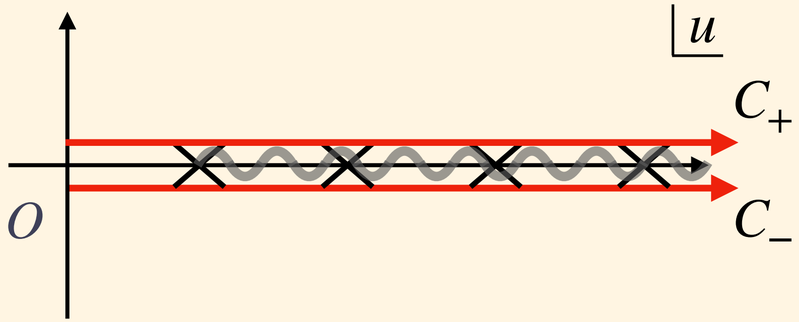
Inv Borel transform

【Method】

- **Conceptual procedure for renormalon subtraction**

Borel transform of $C_1(Q) \underset{\text{PT}}{=} \sum_{n=0}^{\infty} c_n \alpha_s^{n+1}$

$$\Rightarrow B_{C_1}(u) = \sum_{n=0}^{\infty} \frac{c_n}{n!} (u/b_0)^n$$



Avoid divergence by deforming contour

$$C_1(Q) \quad \rightarrow \quad \frac{1}{b_0} \int_{C_\pm} du\, e^{-\frac{u}{b_0 \alpha_s}} B_{C_1}(u)$$

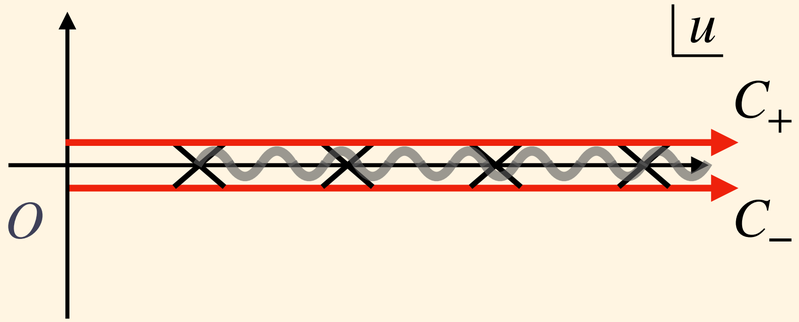
Ambiguous

【Method】

- **Conceptual procedure for renormalon subtraction**

Borel transform of $C_1(Q) \underset{\text{PT}}{=} \sum_{n=0}^{\infty} c_n \alpha_s^{n+1}$

$$\Rightarrow B_{C_1}(u) = \sum_{n=0}^{\infty} \frac{c_n}{n!} (u/b_0)^n$$



Avoid divergence by deforming contour

$$C_1(Q) \quad \rightarrow \quad \frac{1}{b_0} \int_{C_\pm} du\, e^{-\frac{u}{b_0 \alpha_s}} B_{C_1}(u) \quad = \quad [C_1(Q)]_{\text{PV}} \pm \delta C_1(Q)$$

Ambiguous **Finite part** **Ambiguous part**

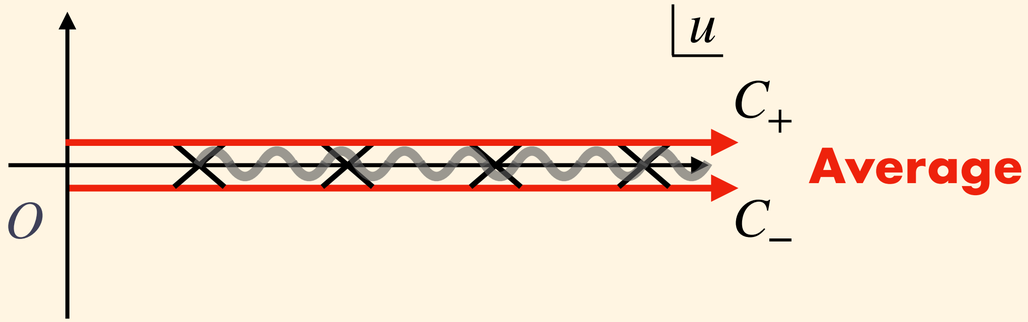
PV : principal value

【Method】

- **Conceptual procedure for renormalon subtraction**

Borel transform of $C_1(Q) \underset{\text{PT}}{=} \sum_{n=0}^{\infty} c_n \alpha_s^{n+1}$

$$\Rightarrow B_{C_1}(u) = \sum_{n=0}^{\infty} \frac{c_n}{n!} (u/b_0)^n$$



Avoid divergence by deforming contour

$$C_1(Q) \quad \rightarrow \quad \frac{1}{b_0} \int_{C_\pm} du\, e^{-\frac{u}{b_0 \alpha_s}} B_{C_1}(u) \quad = \quad \boxed{\begin{aligned} &[C_1(Q)]_{\text{PV}} \pm \delta C_1(Q) \\ &= \frac{1}{b_0} \int_{0,\text{PV}}^{\infty} du\, e^{-\frac{u}{b_0 \alpha_s}} B_{C_1}(u) \end{aligned}}$$

Ambiguous

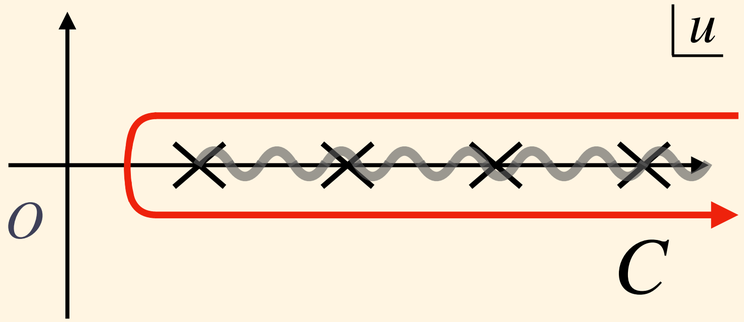
Finite part

【Method】

- **Conceptual procedure for renormalon subtraction**

Borel transform of $C_1(Q) \underset{\text{PT}}{=} \sum_{n=0}^{\infty} c_n \alpha_s^{n+1}$

$$\Rightarrow B_{C_1}(u) = \sum_{n=0}^{\infty} \frac{c_n}{n!} (u/b_0)^n$$



Avoid divergence by deforming contour

$$\left(\int_C du\, e^{-\frac{u}{b_0 \alpha_s}} \frac{1}{1 - u/u_0} \approx e^{-\frac{u_0}{b_0 \alpha_s}} \approx \left(\Lambda_{\text{QCD}}/Q\right)^{2u_0} \right)$$

$$C_1(Q) \quad \rightarrow \quad \frac{1}{b_0} \int_{C_\pm} du\, e^{-\frac{u}{b_0 \alpha_s}} B_{C_1}(u) \quad = \quad [C_1(Q)]_{\text{PV}} \pm \delta C_1(Q)$$

Ambiguous

$$= \pm \frac{1}{2 b_0 i} \int_C du\, e^{-\frac{u}{b_0 \alpha_s}} B_{C_1}(u)$$

Ambiguous part

【Method】

- **Practical problem of PV-prescription by limited knowledge**

$$[C_1(Q)]_{\mathrm{PV}} = \frac{1}{b_0}\int_{0,\mathrm{PV}}^{\infty} du\, e^{-\frac{u}{b_0\alpha_s}} \boxed{B_{C_1}(u)}$$

Definition: all-order sum

$$B_{C_1}(u) = \sum_{n=0}^{\infty} \frac{c_n}{n!}(u/b_0)^n$$

【Method】

- **Practical problem of PV-prescription by limited knowledge**

$$[C_1(Q)]_{\rm PV} = \frac{1}{b_0}\int_{0,\rm PV}^{\infty} du\, e^{-\frac{u}{b_0\alpha_s}} \boxed{B_{C_1}(u)}$$

Definition: all-order sum

$$B_{C_1}(u) = \sum_{n=0}^{\infty} \frac{c_n}{n!}(u/b_0)^n$$

Finite number of coefficients

$$\{c_0, c_1, \cdots, c_k\}$$

Approximation is needed

Recent studies on this direction

2018 Takaura, Kaneko, Kiyo, Sumino → $\alpha_s(M_Z)$
2018 Bazavov, et al → heavy quark masses
2019, 2020 Ayala, Lobregat, Pineda → $\alpha_s(M_Z)$
2020 Takaura

→ Our method utilizes the properties of Fourier transform.

【Method】

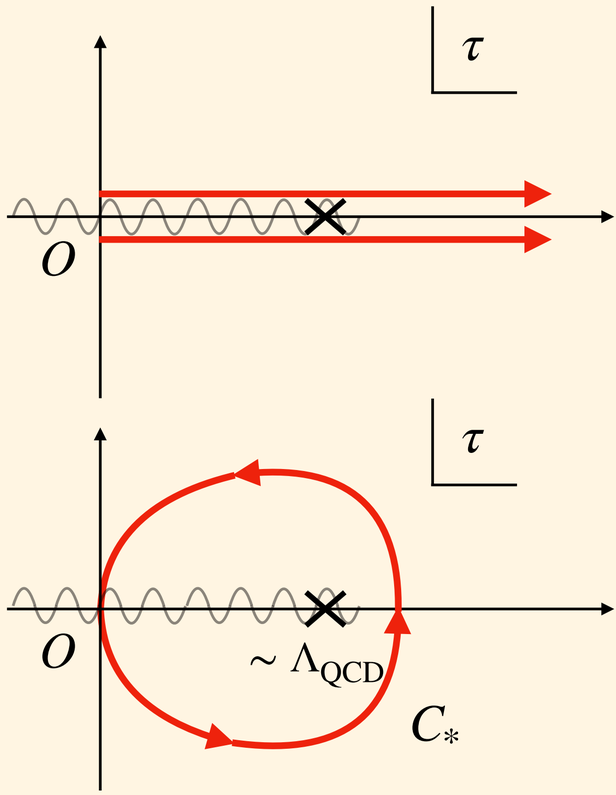
- **Property of our method**

$$C_1(Q) \to [C_1(Q)]_{\mathrm{PV}} \pm \delta C_1(Q)$$

$$[C_1(Q)]_{\mathrm{PV}} : \int_{0,\mathrm{PV}}^{\infty} d\tau \left(\text{Fourier transform of } C_1\right)$$

$$\delta C_1(Q) : \oint_{C_*} d\tau \left(\text{Fourier transform of } C_1\right)$$

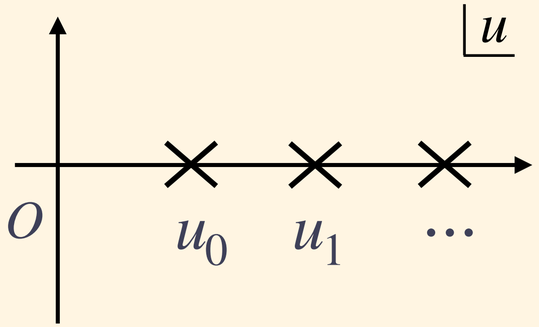
$$\approx N_{u_0}\left(\Lambda_{\mathrm{QCD}}/Q\right)^{2u_0} + N_{u_1}\left(\Lambda_{\mathrm{QCD}}/Q\right)^{2u_1} + \cdots$$



- Incorporating higher order corrections
  → Systematic approximation of the PV-prescription result
- Subtracting multiple renormalons
- Universal ← OPE & RGE

【Method】

- **Property of Fourier transform**



Assumptions: Renormalons of C_1 are at $u = u_0, u_1, \cdots$.

Renormalons are removed in OPE framework.

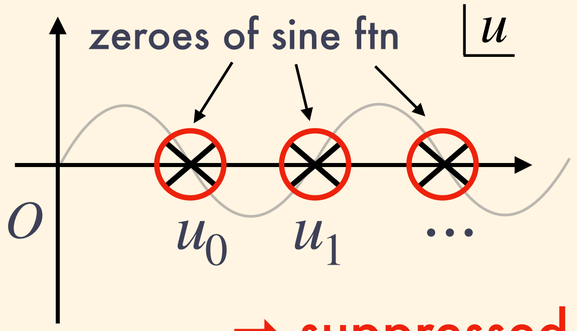
Renormalon structure (apart from normalization)

$$\delta C_1(Q) \approx \left(\Lambda_{\rm QCD}^2/Q^2\right)^{u_0}, \left(\Lambda_{\rm QCD}^2/Q^2\right)^{u_1}, \ldots$$

【Method】

- **Property of Fourier transform**

After FT



→ suppressed

Assumptions: Renormalons of C_1 are at $u = u_0, u_1, \cdots$.

Renormalons are removed in OPE framework.

Renormalon structure (apart from normalization)

$$\delta C_1(Q) \approx \left(\Lambda_{\rm QCD}^2/Q^2\right)^{u_0}, \left(\Lambda_{\rm QCD}^2/Q^2\right)^{u_1}, \ldots$$

$(r = |\vec{x}| = Q^{-a})$

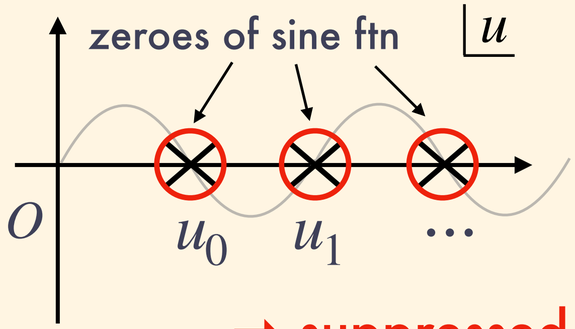
Fourier transform ⟶ $\delta\tilde{C}_1(\tau) = \int d^3\vec{x}\, e^{-i\vec{\tau}\cdot\vec{x}}\, r^{2au'} \delta C_1(r^{-a}) \propto \sin\left(\pi a(u+u')\right)$

Proper choice of parameters (a, u') → suppress the renormalons

【Method】

- **Property of Fourier transform**

After FT



→ suppressed

Assumptions: Renormalons of C_1 are at $u = u_0, u_1, \cdots$.

Renormalons are removed in OPE framework.

Renormalon structure (apart from normalization)

$$\rightarrow \delta C_1(Q) \approx \left(\Lambda_{\rm QCD}^2/Q^2\right)^{u_0}, \left(\Lambda_{\rm QCD}^2/Q^2\right)^{u_1}, \ldots$$

$(r = |\vec{x}| = Q^{-a})$

Fourier transform ⟶ $\delta\tilde{C}_1(\tau) = \int d^3\vec{x}\, e^{-i\vec{\tau}\cdot\vec{x}}\, r^{2au'} \delta C_1(r^{-a}) \propto \sin\left(\pi a(u+u')\right)$

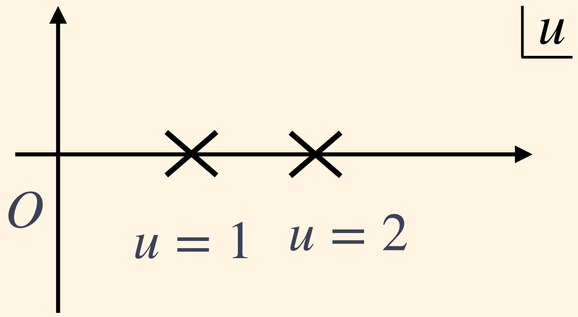
Proper choice of parameters (a, u') → suppress the renormalons

➡ Momentum-space series has better convergence.

【Method】

- **Demonstration using Toy model** *$\beta(\alpha_s)$ is at one loop, and $b_0 = 1$.

$$X(Q) \underset{\text{PT}}{=} \sum_{n=0}^{\infty} n!\left(1-(1/2)^{n+1}\right)\alpha_s(Q)^{n+1}$$

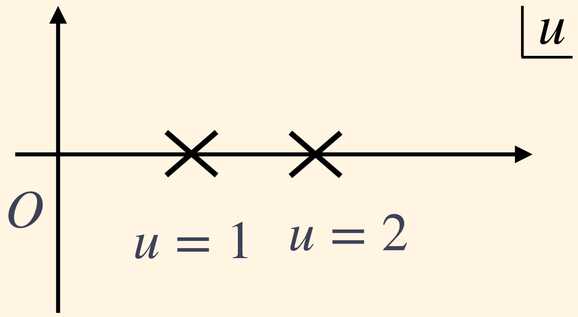


$$B_X(u) = \frac{(\mu^2/Q^2)^u}{(1-u)(2-u)}$$

【Method】

- **Demonstration using Toy model** *$\beta(\alpha_s)$ is at one loop, and $b_0 = 1$.

$$X(Q) \underset{\text{PT}}{=} \sum_{n=0}^{\infty} n!\left(1-(1/2)^{n+1}\right)\alpha_s(Q)^{n+1}$$

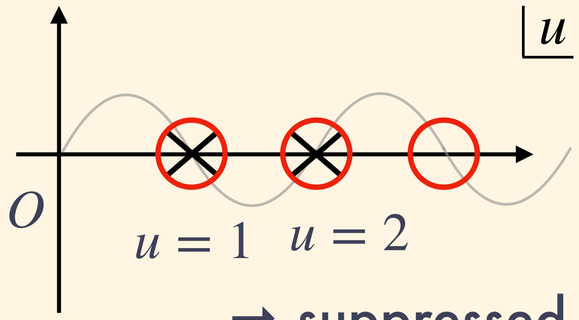


$$B_X(u) = \frac{(\mu^2/Q^2)^u}{(1-u)(2-u)}$$

Fourier transform

$$\tilde{X}(\tau) = \int d^3\vec{x}\, e^{-i\vec{\tau}\cdot\vec{x}}\, \frac{X(1/r)}{r^2} \quad (r = |\vec{x}| = 1/Q)$$

$$(a, u') = (1, -1)$$



→ suppressed (eliminated)

【Method】

$$B_X(u) = \frac{(\mu^2/Q^2)^u}{(1-u)(2-u)}$$

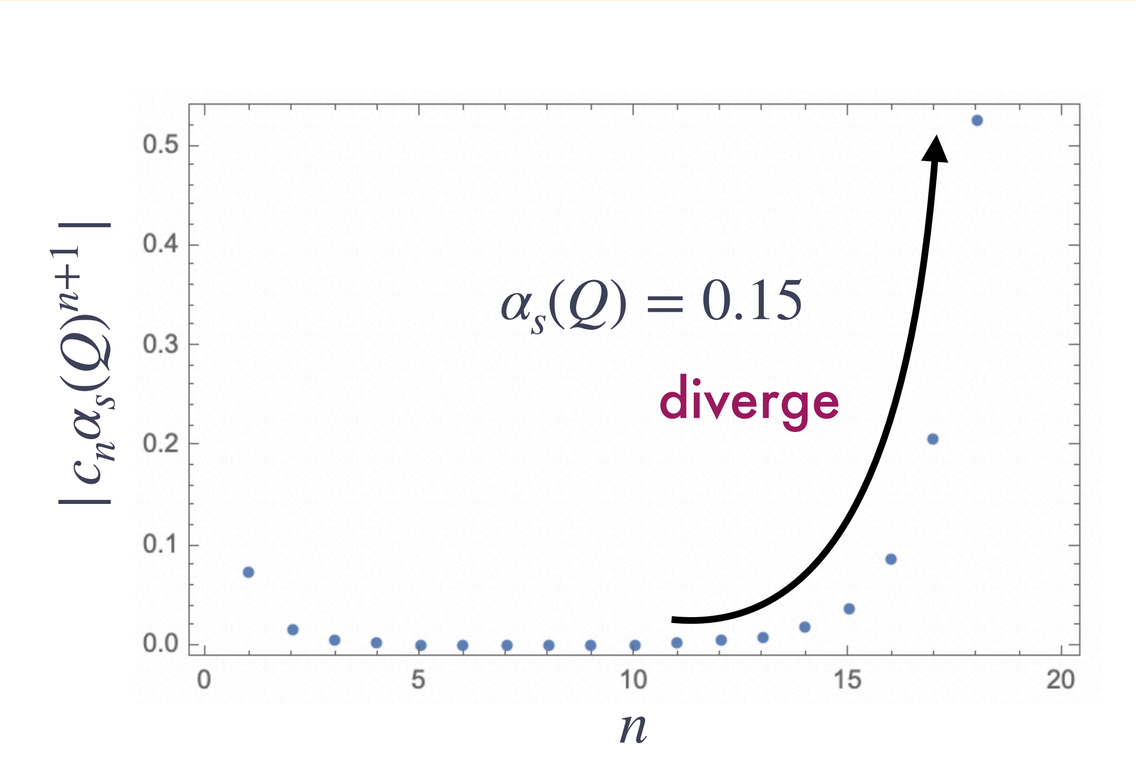
- **Demonstration using Toy model**

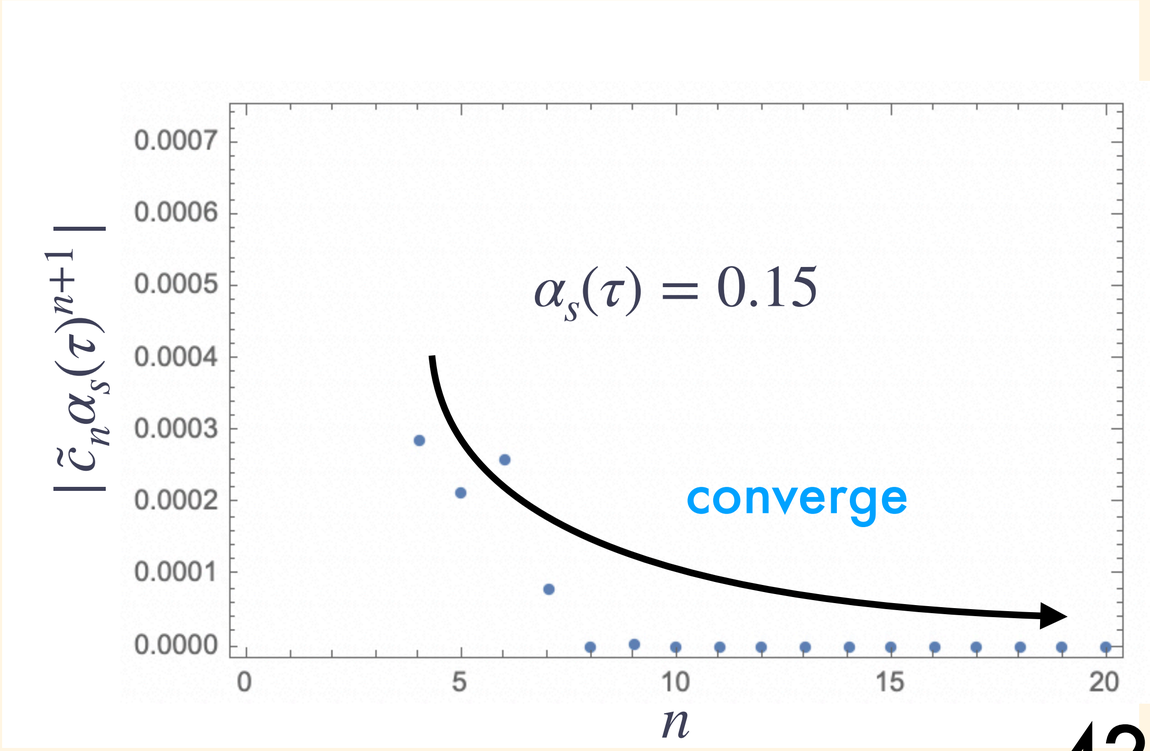
*$\beta(\alpha_s)$ is at one loop, and $b_0 = 1$.

$$X(Q) \underset{\text{PT}}{=} \sum_{n=0}^{\infty} n!\left(1-(1/2)^{n+1}\right)\alpha_s(Q)^{n+1}$$

$$\tilde{X}(\tau) = \int d^3\vec{x}\, e^{-i\vec{\tau}\cdot\vec{x}}\, \frac{X(1/r)}{r^2} \quad (r = |\vec{x}| = 1/Q)$$

$$(a, u') = (1, -1)$$





【Method】

- **Convergence of momentum series → separate renormalons**

$$\tilde{C}_1(\tau) = \int d^3\vec{x}\, e^{-i\vec{\tau}\cdot\vec{x}}\, r^{2au'} C_1(r^{-a})$$

$$= \tilde{c}_0 \alpha_s(\mu) + \tilde{c}_1(L_\tau)\alpha_s(\mu)^2 + \tilde{c}_2(L_\tau)\alpha_s(\mu)^3 + \cdots$$

$; L_\tau = \log\left(\mu^2/\tau^{2a}\right)$ → diverge in the small τ region

【Method】

- **Convergence of momentum series → separate renormalons**

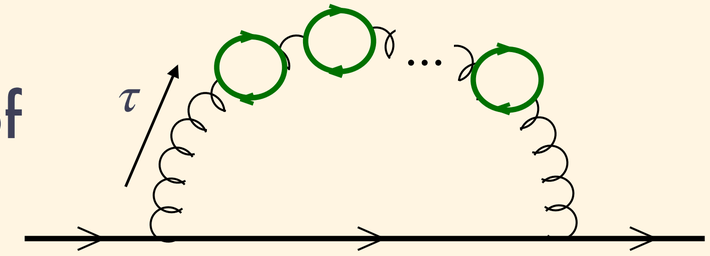
$$\tilde{C}_1(\tau) = \int d^3\vec{x}\, e^{-i\vec{\tau}\cdot\vec{x}}\, r^{2au'} C_1(r^{-a})$$

$$= \tilde{c}_0\alpha_s(\mu) + \tilde{c}_1(L_\tau)\alpha_s(\mu)^2 + \tilde{c}_2(L_\tau)\alpha_s(\mu)^3 + \cdots$$

$; L_\tau = \log\left(\mu^2/\tau^{2a}\right)$ → diverge in the small τ region

Inverse transform (momentum integral) → revives the divergence from renormalons

↔ Integrand of



$\sim f(\tau^2)\left[-\log(\tau^2)\right]^n b_0^n \alpha_s^{n+1}$

【Method】

- **Convergence of momentum series → separate renormalons**

Inverse transform (momentum integral) → revives the divergence from renormalons

$$\tilde{C}_1(\tau) = \int d^3\vec{x}\, e^{-i\vec{\tau}\cdot\vec{x}}\, r^{2au'} C_1(r^{-a})$$

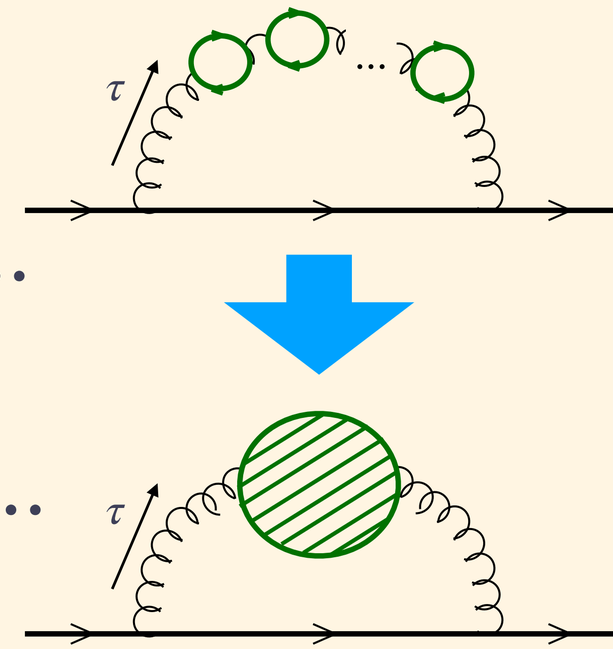
$L_\tau = \log(\mu^2/\tau^{2a})$

$$= \tilde{c}_0\alpha_s(\mu) + \tilde{c}_1(L_\tau)\alpha_s(\mu)^2 + \tilde{c}_2(L_\tau)\alpha_s(\mu)^3 + \cdots$$

$\mu = \tau^a$

(RG improvement)

$$= \tilde{c}_0\alpha_s(\tau^a) + \tilde{c}_1(0)\alpha_s(\tau^a)^2 + \tilde{c}_2(0)\alpha_s(\tau^a)^3 + \cdots$$



$$\alpha_s(\tau^a) \approx \frac{\alpha_s(\mu)}{1 - b_0\alpha_s(\mu)L_\tau}$$: resums higher L_τ terms

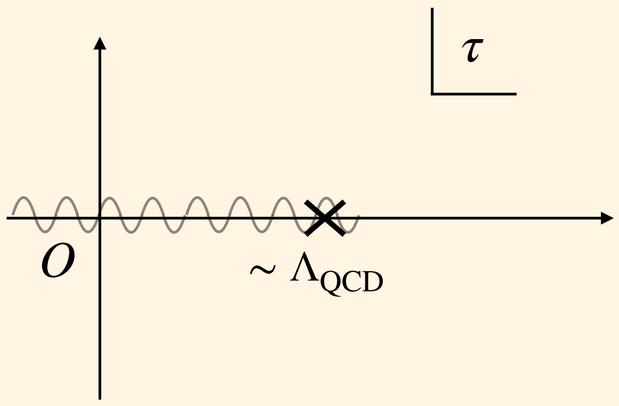
【Method】

- **Convergence of momentum series → separate renormalons**

$$\text{Inverse FT} = \frac{r^{-2au'-1}}{2\pi^2}\int_0^\infty d\tau\,\tau\sin(\tau r)\,\tilde{C}_1(\tau)$$

Divergent

$$\tilde{C}_1(\tau) = \tilde{c}_0\alpha_s(\tau^a) + \tilde{c}_1(0)\alpha_s(\tau^a)^2 + \tilde{c}_2(0)\alpha_s(\tau^a)^3 + \cdots$$



Singularity from $\alpha_s(\tau^a)$

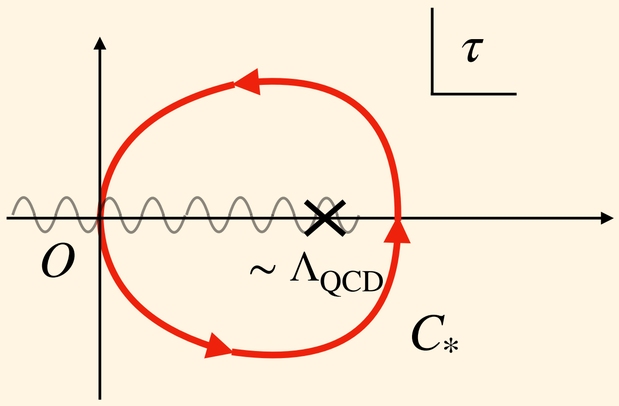
【Method】

- **Convergence of momentum series → separate renormalons**

$$\text{Inverse FT} = \frac{r^{-2au'-1}}{2\pi^2}\int_0^\infty d\tau\,\tau\sin(\tau r)\,\tilde{C}_1(\tau)$$

Divergent

$$\tilde{C}_1(\tau) = \tilde{c}_0\alpha_s(\tau^a) + \tilde{c}_1(0)\alpha_s(\tau^a)^2 + \tilde{c}_2(0)\alpha_s(\tau^a)^3 + \cdots$$



Singularity from $\alpha_s(\tau^a)$

Separating renormalons

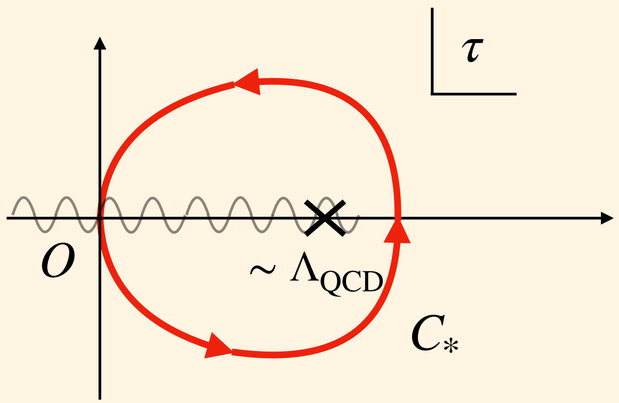
$$\longrightarrow \quad \delta C_1(Q) = \frac{1}{2}\frac{r^{-2au'-1}}{2\pi^2}\int_{C_*} d\tau\,\tau\sin(\tau r)\,\tilde{C}_1(\tau)$$

【Method】

- **Convergence of momentum series → separate renormalons**

$$\text{Inverse FT} = \frac{r^{-2au'-1}}{2\pi^2}\int_0^\infty d\tau\,\tau\sin(\tau r)\,\tilde{C}_1(\tau)$$

Divergent



Singularity from $\alpha_s(\tau^a)$

$$\tilde{C}_1(\tau) = \tilde{c}_0\alpha_s(\tau^a) + \tilde{c}_1(0)\alpha_s(\tau^a)^2 + \tilde{c}_2(0)\alpha_s(\tau^a)^3 + \cdots$$

Separating renormalons

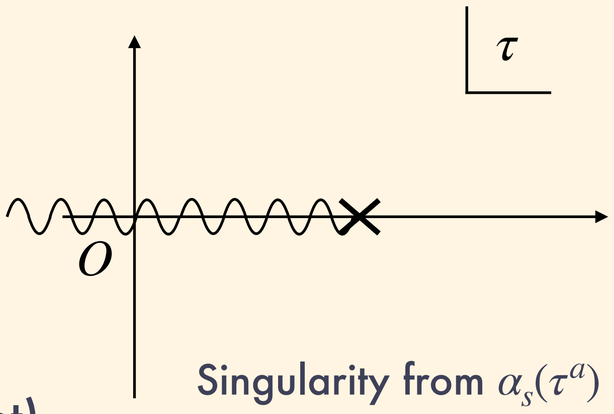
$$\longrightarrow \quad \delta C_1(Q) = \frac{1}{2}\frac{r^{-2au'-1}}{2\pi^2}\int_{C_*} d\tau\,\tau\,\underline{\sin(\tau r)}\,\tilde{C}_1(\tau)$$

Taylor-expansion $(\tau r \ll 1)$

$$\approx N_0\left(\Lambda_{\rm QCD}^2/Q^2\right)^{u_0} + N_1\left(\Lambda_{\rm QCD}^2/Q^2\right)^{u_1} + \ldots$$

【Method】

- **Renormalon-subtracted representation**



Singularity from $\alpha_s(\tau^a)$

$$C_1(Q) = c_0\alpha_s(Q) + c_1\alpha_s(Q)^2 + \cdots$$

Divergent

Fourier transform
(+ RG improvement)

$$\tilde{C}_1(\tau) = \tilde{c}_0\alpha_s(\tau^a) + \tilde{c}_1\alpha_s(\tau^a)^2 + \cdots$$

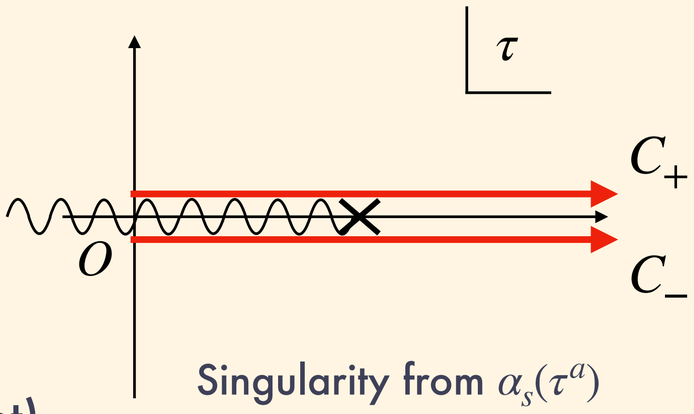
Convergent

: singular due to $\alpha_s(\tau^a)$

Momentum integral revives divergence.

【Method】

- **Renormalon-subtracted representation**



$$C_1(Q) = c_0\alpha_s(Q) + c_1\alpha_s(Q)^2 + \cdots$$

Divergent

Fourier transform
(+ RG improvement)

$$\tilde{C}_1(\tau) = \tilde{c}_0\alpha_s(\tau^a) + \tilde{c}_1\alpha_s(\tau^a)^2 + \cdots$$

Convergent

: singular due to $\alpha_s(\tau^a)$

Momentum integral revives divergence.

Contour-deformed inv FT

$$\frac{r^{-2au'-1}}{2\pi^2}\int_{C_\pm} d\tau\,\tau\sin(\tau r)\,\tilde{C}_1(\tau)$$
$$= \left[C_1(Q)\right]_{\rm PV} \pm \delta C_1(Q)$$

$$\left[C_1(Q)\right]_{\rm PV} = \frac{r^{-2au'-1}}{2\pi^2}\int_{0,\rm PV}^{\infty} d\tau\,\tau\sin(\tau r)\,\tilde{C}_1(\tau)\,, \quad \delta C_1(Q) = \frac{r^{-2au'-1}}{4\pi^2}\int_{C_*} d\tau\,\tau\sin(\tau r)\,\tilde{C}_1(\tau)$$

【Method】

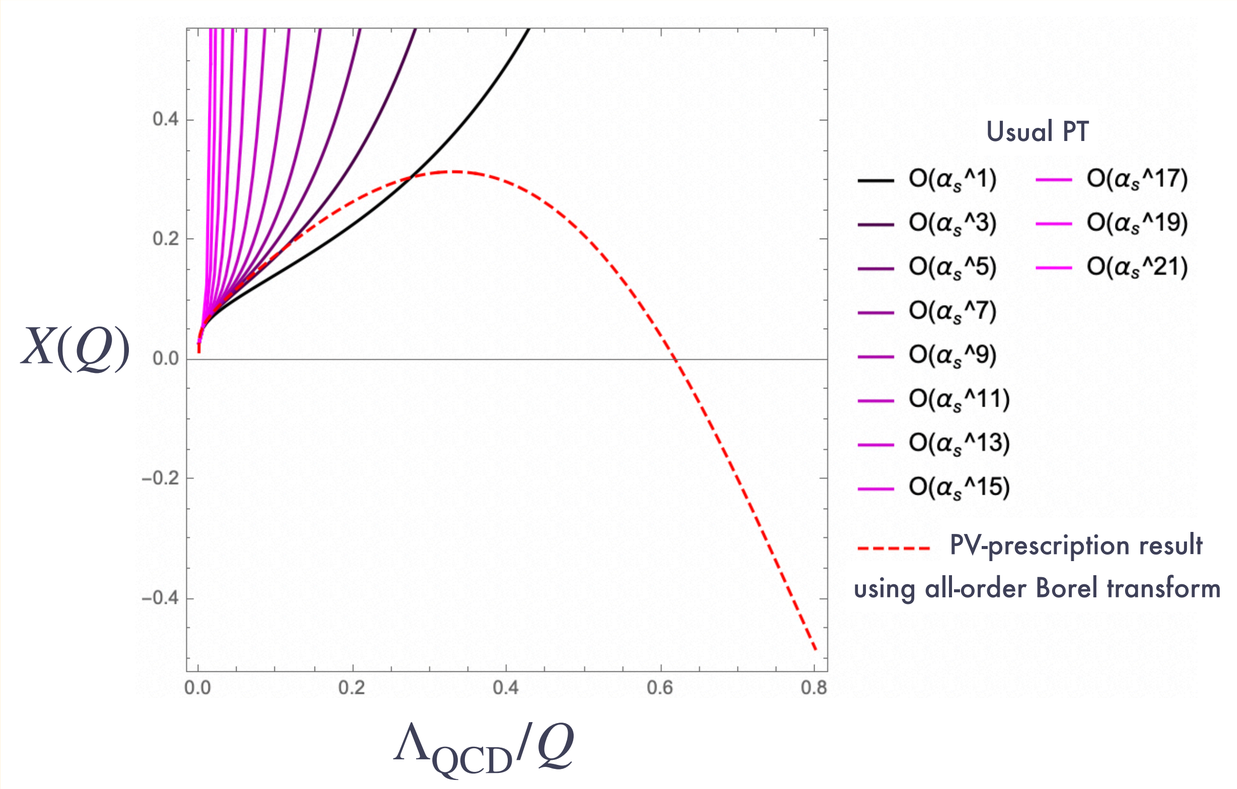
- **Demonstration of Renormalon subtraction using Toy model**

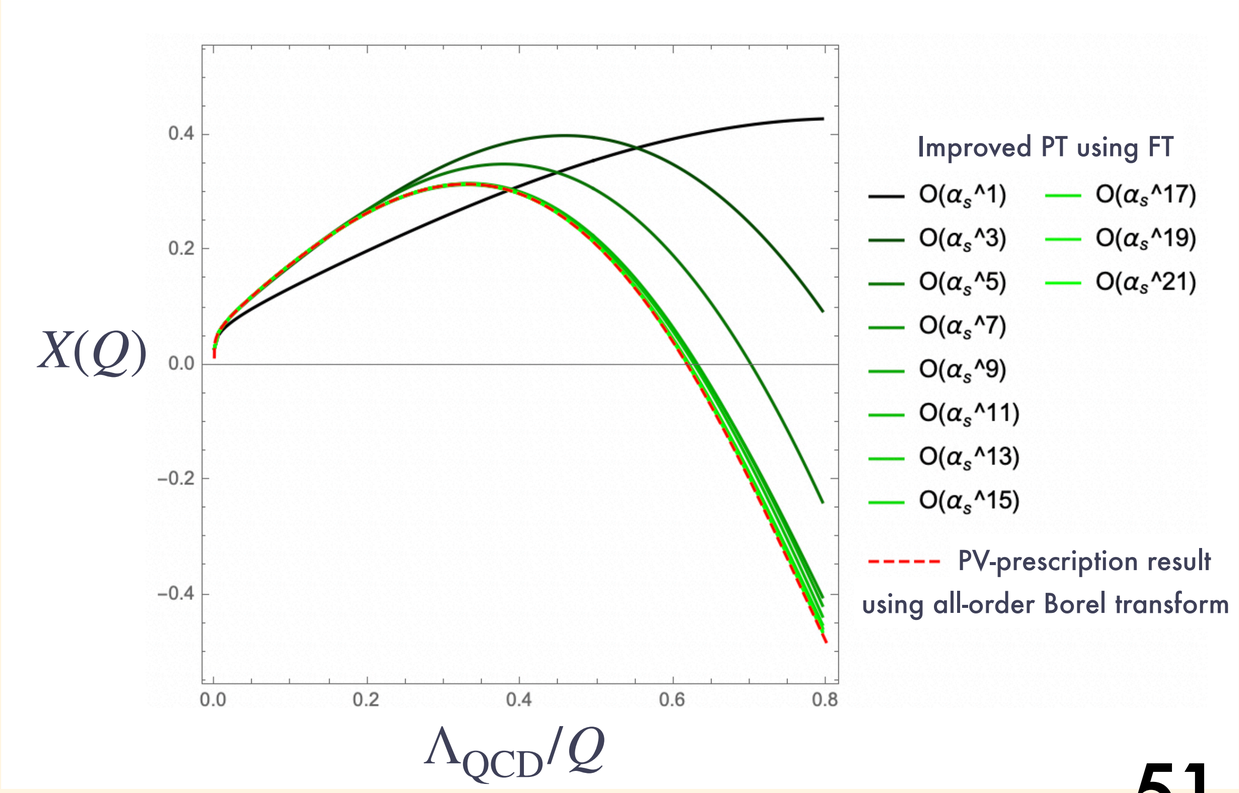
$$B_X(u) = \frac{(\mu^2/Q^2)^u}{(1-u)(2-u)} \rightarrow \boxed{[X(Q)]_{\rm PV} = \int_{0,\rm PV}^{\infty} du\, e^{-\frac{u}{\alpha_s}} B_X(u)}$$

All order

$$\boxed{[X(Q)]_{\rm PV}^{(k)} = \frac{1}{2\pi^2 Q}\int_{0,\rm PV}^{\infty} d\tau\, \tau \sin(\tau/Q)\, \tilde{X}(\tau)^{(k)}}$$

Approximation





$\beta(\alpha_s)$ is at one loop, and $b_0 = 1$.

Table of contents

1. **Introduction: Ambiguity in QCD — renormalon problem**

   → *How to separate renormalons from Wilson coefficient ?*

2. **Method: Renormalon subtraction using Fourier transform**

   → *Fourier transform + RG improvement in momentum space*

3. **Application: B, D meson mass**

4. **Conclusions**

【Application】

- **Renormalon subtraction for B, D meson mass**

$H = B, D$
$h = b, c$

$$[\overline{M}_H]_{\rm OPE} = \left[\frac{M_H + 3M_{H^*}}{4}\right]_{\rm OPE} = m_h + \overline{\Lambda} + \frac{\mu_\pi^2}{2m_h} + O(1/m_h^2)$$

↓ Our method (suppress $u = 1/2, 1$ renormalon)

$$[\overline{M}_H]_{\rm OPE} = [m_h]_{\rm PV} + [\overline{\Lambda}]_{\rm PV} + \frac{[\mu_\pi^2]_{\rm PV}}{2m_h} + O\left(\frac{\Lambda_{\rm QCD}^3}{m_h^2}\right)$$

Subtracting multiple renormalons ⟶ Determine non-PT matrix elements

【Application】

- **Procedure**

$$\boxed{\left[\overline{M}_H\right]_{\rm OPE} = \left[m_h\right]_{\rm PV} + \left[\overline{\Lambda}\right]_{\rm PV} + \frac{\left[\mu_\pi^2\right]_{\rm PV}}{2m_h} + O\left(\frac{\Lambda_{\rm QCD}^3}{m_h^2}\right)}$$

Taking PV of momentum integral $\longrightarrow$ Subtract $O(\Lambda_{\rm QCD}), O(\Lambda_{\rm QCD}^2)$ ambiguities

$$m_h = \bar{m}_h\left(1 + d_0\alpha_s + d_1\alpha_s^2 + d_2\alpha_s^3 + d_3\alpha_s^4\right) \quad O(\alpha_s^4)$$

- Experimental inputs : $\left[\overline{M}_B\right]_{\rm Exp} = 5.313\,{\rm GeV}, \left[\overline{M}_D\right]_{\rm Exp} = 1.971\,{\rm GeV}$
- Other inputs : $\bar{m}_b = 4.18\,^{+0.03}_{-0.02}\,{\rm GeV},\ \bar{m}_c = 1.27(2)\,{\rm GeV},$
  (from PDG)
  $\alpha_s(M_Z) = 0.1179(10)$

【Application】

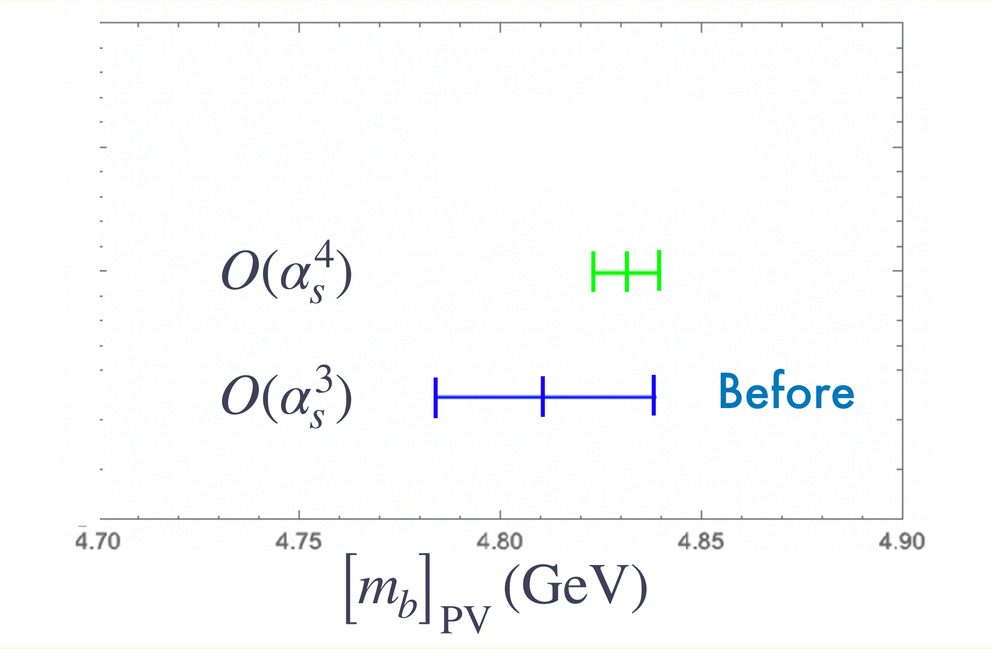
- **Scale dependence of $[m_b]_{\rm PV}$ & $[m_c]_{\rm PV}$**

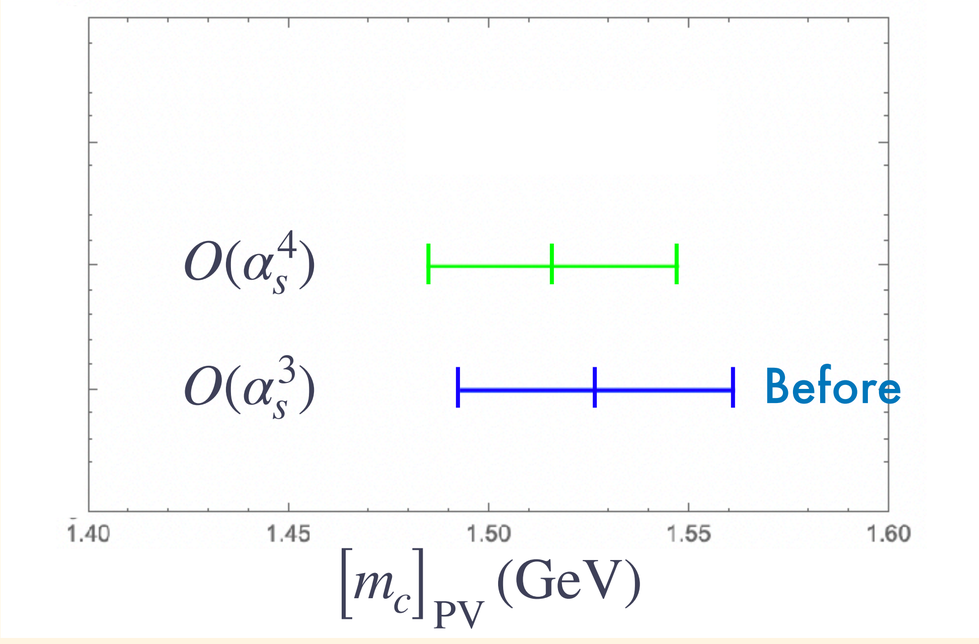
$O(\alpha_s^4)$

$$[m_b]_{\rm PV} = 4.831(8)\,\text{GeV}, \quad [m_c]_{\rm PV} = 1.516(31)\,\text{GeV}$$

(for $\bar{m}_b = 4.18\,\text{GeV}$) (for $\bar{m}_c = 1.27\,\text{GeV}$)

(#) : error from scale variation within $\mu \in [\tau^2/2, 2\tau^2]$





【Application】

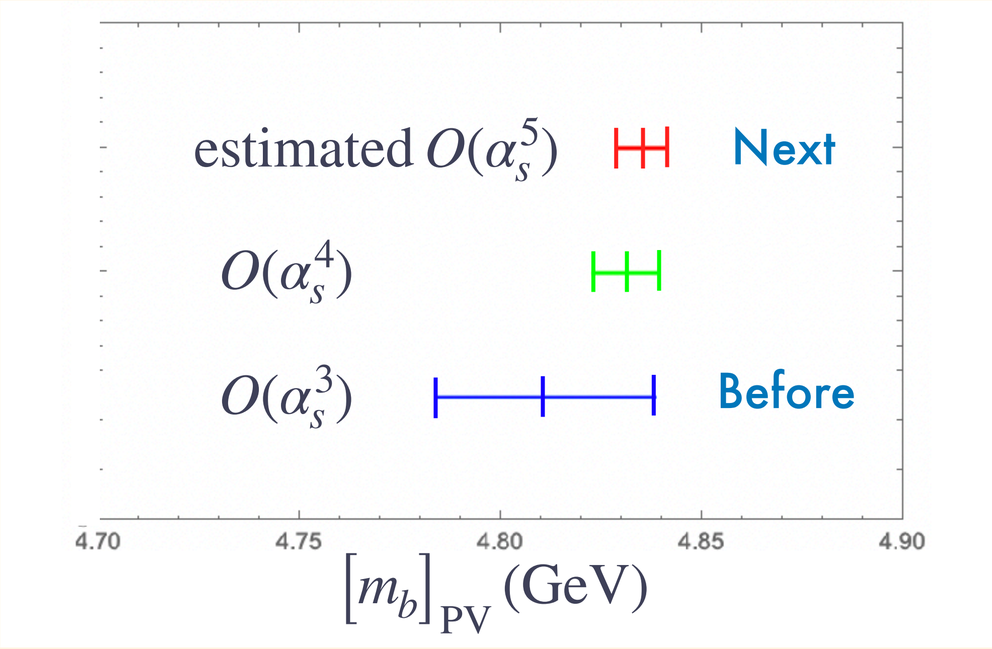
- **Scale dependence of $[m_b]_{\rm PV}$ & $[m_c]_{\rm PV}$**

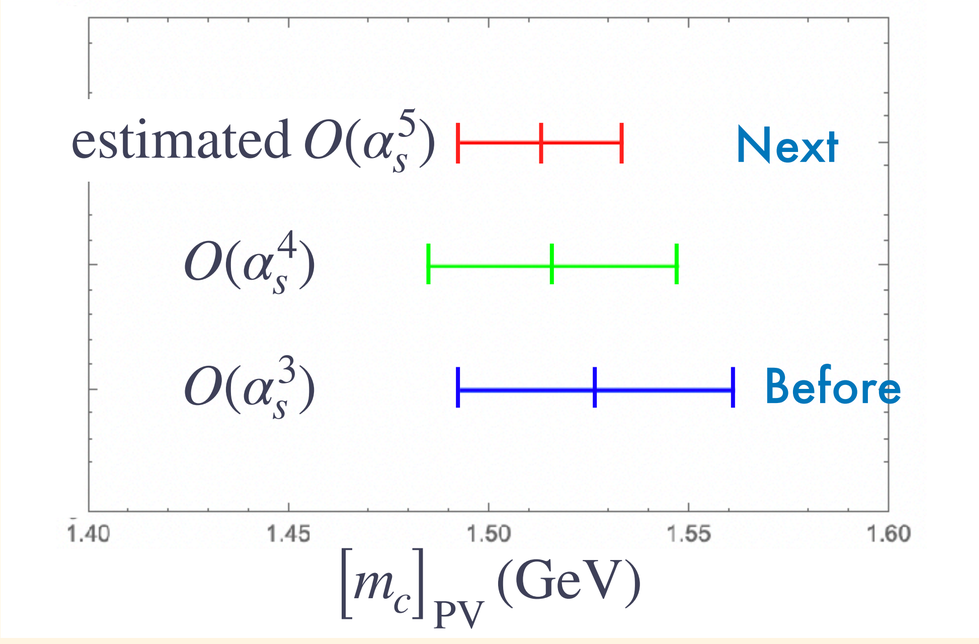
$O(\alpha_s^4)$

$$[m_b]_{\rm PV} = 4.831(8)\,\text{GeV}, \quad [m_c]_{\rm PV} = 1.516(31)\,\text{GeV}$$

(for $\bar{m}_b = 4.18\,\text{GeV}$) (for $\bar{m}_c = 1.27\,\text{GeV}$)

(#) : error from scale variation within $\mu \in [\tau^2/2, 2\tau^2]$





PT error gets smaller as higher order correction is incorporated.

【Application】

- **Determination of $[\overline{\Lambda}]_{\rm PV}$ & $[\mu_\pi^2]_{\rm PV}$**

$O(\alpha_s^4)$

$$[\overline{\Lambda}]_{\rm PV} = 0.495\,(15)_\mu(49)_{\bar{m}_b}(12)_{\bar{m}_c}(13)_{\alpha_s}\ \text{GeV}$$

$$[\mu_\pi^2]_{\rm PV} = -\,0.12\,(13)_\mu(15)_{\bar{m}_b}(11)_{\bar{m}_c}(4)_{\alpha_s}\ \text{GeV}^2$$

- $[\overline{\Lambda}]_{\rm PV}$: PT error is sufficiently small due to renormalon subtraction.

  (If renormalons remain, $\delta\overline{\Lambda} \sim O(\Lambda_{\rm QCD})$)

- $[\mu_\pi^2]_{\rm PV}$: Why is PT error still large?

【Application】

- **Determination of $[\overline{\Lambda}]_{\rm PV}$ & $[\mu_\pi^2]_{\rm PV}$**

$O(\alpha_s^4)$

$$[\overline{\Lambda}]_{\rm PV} = 0.495\,(15)_\mu(49)_{\bar{m}_b}(12)_{\bar{m}_c}(13)_{\alpha_s}\,{\rm GeV}$$

$$[\mu_\pi^2]_{\rm PV} = -\,0.12\,(13)_\mu(15)_{\bar{m}_b}(11)_{\bar{m}_c}(4)_{\alpha_s}\,{\rm GeV}^2$$

- $[\overline{\Lambda}]_{\rm PV}$: PT error is sufficiently small due to renormalon subtraction.

(If renormalons remain, $\delta\overline{\Lambda} \sim O(\Lambda_{\rm QCD})$)

- $[\mu_\pi^2]_{\rm PV}$: Why is PT error still large?

$\rightarrow$ estimated $O(\alpha_s^5)$ analysis $\quad [\mu_\pi^2]_{\rm PV} = -\,0.09\,(7)_\mu\,{\rm GeV}^2$

Next order perturbation will improve the PT accuracy.

【Application】

- **Previous studies on $[\overline{\Lambda}]_{\rm PV}$ & $[\mu_\pi^2]_{\rm PV}$ → consistent**

Our result

$$[\overline{\Lambda}]_{\rm PV} = 0.495\,(15)_\mu(49)_{\bar{m}_b}(12)_{\bar{m}_c}(13)_{\alpha_s}\,{\rm GeV} = 0.495\,(53)\,{\rm GeV},$$
$$[\mu_\pi^2]_{\rm PV} = -\,0.12\,(13)_\mu(15)_{\bar{m}_b}(11)_{\bar{m}_c}(4)_{\alpha_s}\,{\rm GeV}^2 = -\,0.12\,(23)\,{\rm GeV}^2$$

Subtract $O(\Lambda_{\rm QCD})$ renormalon ambiguity → determine $\overline{\Lambda}$ only

Ayala, Lobreget, Pineda, 1909.01370

$$\bar{\Lambda}_{\rm PV} = 477(\mu)^{-8}_{+17}(Z_m)^{+11}_{-12}(\alpha_s)^{-8}_{+9}(\mathcal{O}(1/m_h))^{+46}_{-46}\ {\rm MeV}\,,$$

Subtract $O(\Lambda_{\rm QCD})$ renormalon ambiguity → determine $\overline{\Lambda}$ and μ_π^2

Bazavov *et al*, 1802.04248

$$\bar{\Lambda}_{\rm PV} = 0.435(31)\ {\rm GeV}\,, \quad (\mu_\pi^2)_{\rm PV} = 0.05(22)\ {\rm GeV}^2\,.$$

【Application】

- **On-going work: Precise determination of $|V_{cb}|$**

$[\mu_\pi^2]_{\mathrm{PV}}$ → Other observable: Inclusive semileptonic decay width of B meson

$$\Gamma(B \to X_c \ell \nu)$$

Experiment ⟷ $\Gamma_{th} = \dfrac{G_F^2 |V_{cb}|^2}{192\pi^3} m_b^5 A_{EW} \left[\text{QCD corrections}\right]$

$$\text{QCD corrections} \underset{\text{OPE}}{=} \left[g_0 + g_1\alpha_s + g_2\alpha_s^2 + \cdots\right] + \left[C_{kin}\frac{\mu_\pi^2}{m_b^2} + C_{cm}\frac{\mu_G^2}{m_b^2} + \cdots\right]$$

$O(\alpha_s^3)$

Renormalon subtraction + Using well-defined non-PT parameter $[\mu_\pi^2]_{\mathrm{PV}}$

→ Precise determination of $|V_{cb}|$

【Conclusions】

- Subtracting the renormalons in the Wilson coefficient in the framework of OPE

  → Improved non-perturbative matrix elements

- A new method for separating renormalons, which utilizes the properties of FT

  - Fourier transform + RG improvement → convergent series in momentum space.
  - Equivalent to PV prescription at large order of perturbation.

- B, D meson mass → Improved non-PT matrix elements of order $O(\Lambda_{\rm QCD}), O(\Lambda_{\rm QCD}^2)$

$$\left[\overline{\Lambda}\right]_{\rm PV} = 0.495\,(53)\,{\rm GeV}, \qquad \left[\mu_\pi^2\right]_{\rm PV} = -\,0.12\,(23)\,{\rm GeV}^2,$$

  - Incorporating the next order PT correction → PT error will shrink.
  - Can be used to precise computation of other observable.

Back up slides

【Back-up】

- **Universality of Non-PT matrix elements**

HQET : 1 heavy quark + light sector → 2 heavy quark + light sector

(bottom, charm)

∵ Renormalon is sensitive to light sector $c_n \sim b_0^n n!\,,$ $b_0 = \frac{1}{4\pi}(11 - \frac{2}{3}n_l)$

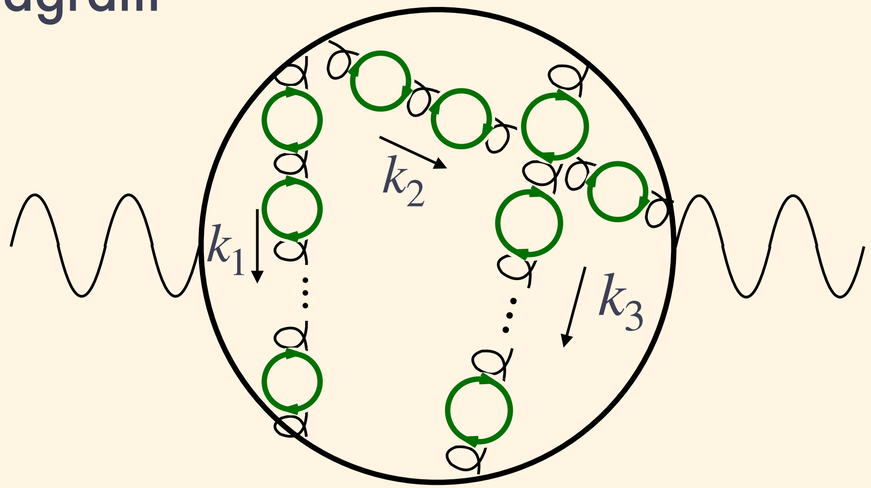
→ Structure of low energy physics should be same

in order to subtract renormalons of different quarks.

【Back-up】

- **Momentum integral representation**

Typical renormalon diagram



→ Very hard to construct one-parameter integral representation.

【Back-up】

- **Remove ambiguities with anomalous dim & α_s corrections**

$$\delta X(Q) = N_{u_*}\left(\Lambda_{\overline{\mathrm{MS}}}/Q\right)^{2u_*}\alpha_s^{\gamma_0/b_0}(1+\mathcal{O}(\alpha_s))$$

1993 Beneke

$$= N_{u_*}\left(\Lambda_{\overline{\mathrm{MS}}}/Q\right)^{2u_*}(1+q_1\log(1/Q^2)+q_2\log(1/Q^2)^2+\cdots)$$

We find Fourier transform so that $\delta\tilde{X}(\tau)$ doesn't have $\log(\tau)$ corrections.

(Detailed discussion: 2106.03687)

【Back-up】

- **Demonstration of Renormalon subtraction using Toy model**

$$X(Q) = \sum_{n=0}^{20} n!\left(1-(1/2)^{n+1}\right)\alpha_s(Q)^{n+1}$$
$$= \sum_{n=0}^{20} c_n \alpha_s(Q)^{n+1}$$

n	c_n	n	c_n	n	c_n
0	0.5	7	$5.02031 * 10^3$	14	$8.71756 * 10^{10}$
1	0.75	8	$4.02413 * 10^4$	15	$1.30765 * 10^{12}$
2	1.75	9	$3.62526 * 10^5$	16	$2.09226 * 10^{13}$
3	5.625	10	$3.62703 * 10^6$	17	$3.55686 * 10^{14}$
4	23.25	11	$3.99071 * 10^7$	18	$6.40236 * 10^{15}$
5	$1.18125 * 10^2$	12	$4.78943 * 10^8$	19	$1.21645 * 10^{17}$
6	$7.14375 * 10^2$	13	$6.22664 * 10^9$	20	$2.4329 * 10^{18}$

$$\tilde{X}(\tau) = \int d^3\vec{x}\, e^{i\vec{\tau}\cdot\vec{x}} \frac{X(1/r)}{r^2} \qquad (r = |\vec{x}| = 1/Q)$$
$$= \frac{4\pi}{\tau}\sum_{n=0}^{20} \tilde{c}_n \alpha_s(\tau)^{n+1}$$

n	$\tilde{c}_n$	n	$\tilde{c}_n$	n	$\tilde{c}_n$
0	0.785398	7	-4.88026	14	$-6.48891 * 10^4$
1	1.93213	8	$-1.42937 * 10^2$	15	$1.88058 * 10^5$
2	2.23991	9	$-1.95803 * 10^2$	16	$2.80674 * 10^6$
3	0.564347	10	$-1.20629 * 10^3$	17	$1.87057 * 10^7$
4	2.80825	11	$-8.69792 * 10^3$	18	$8.78724 * 10^7$
5	22.7583	12	$-2.99181 * 10^4$	19	$3.30672 * 10^8$
6	46.6463	13	$-5.86192 * 10^4$	20	$1.00739 * 10^9$

$\beta(\alpha_s)$ is at one loop, and $b_0 = 1$.