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Aspects of Holographic Entanglement of Purification Conjecture

Koji Umemoto (YITP)

Collaborators

Arpan Bhattacharyya (YITP)
Pawel Caputa (YITP)
Alexander Jahn (Freie U.)
Masamichi Miyaji (U. of Tokyo)
Tadashi Takayanagi (YITP)
Yang Zhou (Fudan U.)

Plan of talk

Introduction

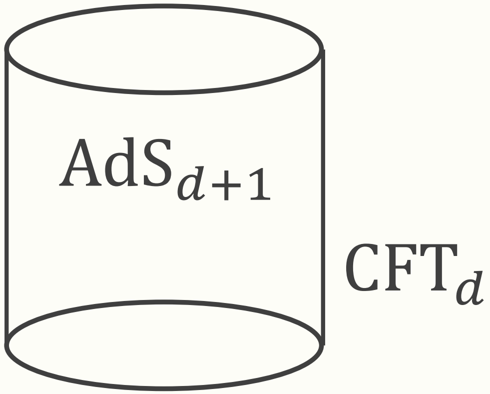
AdS/CFT correspondence

Holographic principle

A way to formulate Quantum Gravity

AdS/CFT correspondence

[Maldacena '97] [Gubser-Klebanov-Polyakov, Witten '98]...



Penrose diagram

Quantum gravity on $d+1$-dimensional asymptotic Anti de Sitter (AdS) spacetime

Equivalent

d-dimensional Conformal Field Theory (CFT) on the boundary

AdS/CFT correspondence

Ultimate goal

Reveal the mechanism of holography from AdS/CFT

AdS/CFT correspondence

An approach

Application of quantum information theory

Ryu-Takayanagi formula

[Ryu-Takayanagi '06]

$$S_A := -\mathrm{Tr}\rho_A \log \rho_A = \min_{\gamma_A} \frac{\mathrm{Area}(\gamma_A)}{4G_N}$$

Entanglement entropy (CFT)　　Area of minimal surface (AdS)

Ryu-Takayanagi formula

Entanglement entropy

$$S(\rho_A) \coloneqq -\mathrm{Tr}\rho_A \log \rho_A \qquad (\rho_A \coloneqq \mathrm{Tr}_B[|\Psi\rangle\langle\Psi|_{AB}])$$

A measure of quantum entanglement on $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ (pure states)

$$|\Psi\rangle_{AB} = \textstyle\sum_i \sqrt{\lambda_i}\, |i\rangle_A |i\rangle_B$$

Schmidt decomposition

$$S(\rho_A) = -\textstyle\sum_i \lambda_i \log \lambda_i$$

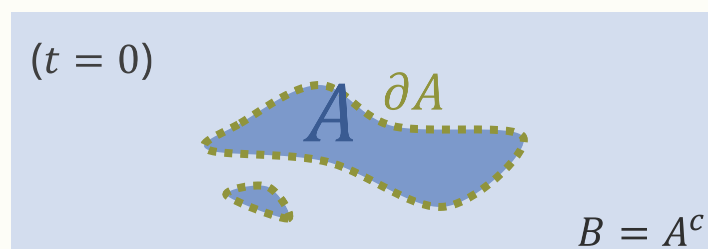
$$(\rho_A = \textstyle\sum_i \lambda_i\, |i\rangle\langle i|_A)$$

- An operational interpretation in many copies regime

$$|\Psi\rangle_{AB} \rightleftarrows |\mathrm{EPR}\rangle_{AB}^{\otimes S_A}$$

Local Operations and Classical Communication

- In quantum field theory A is a spatial subregion



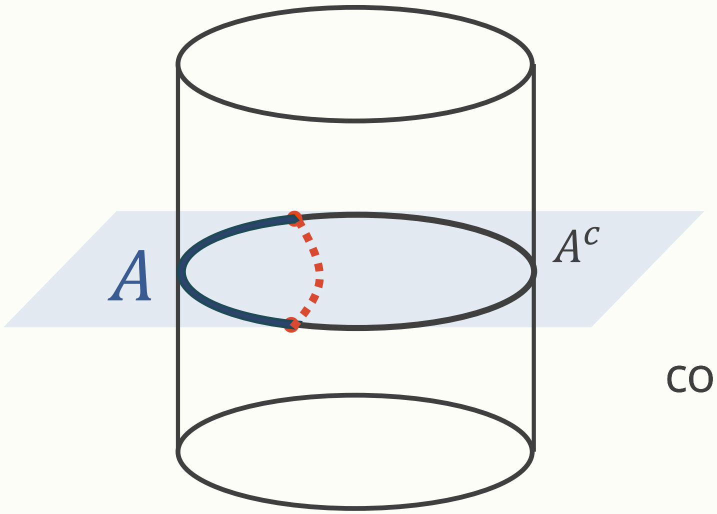
Ryu-Takayanagi formula

Holographic entanglement entropy

$$\min_{\gamma_A} \frac{\mathrm{Area}(\gamma_A)}{4G_N}$$

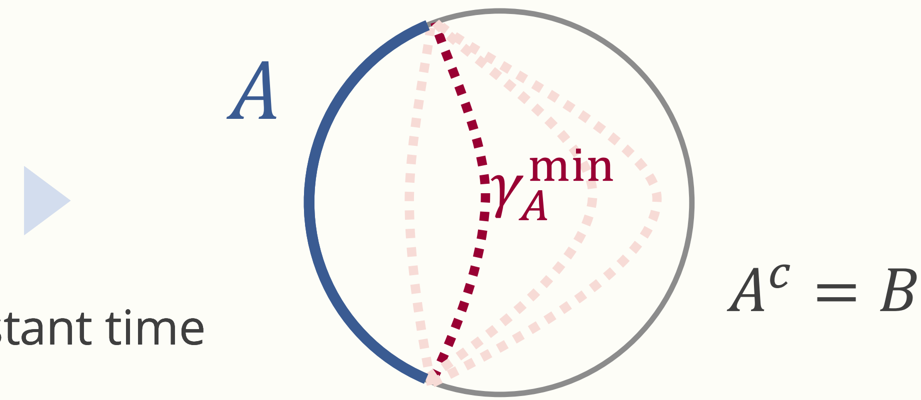
The minimal area of codimension-2 surfaces which satisfy

(1) $\partial\gamma_A = \partial A$ (2) γ_A is homologous to A



Global pure AdS

constant time

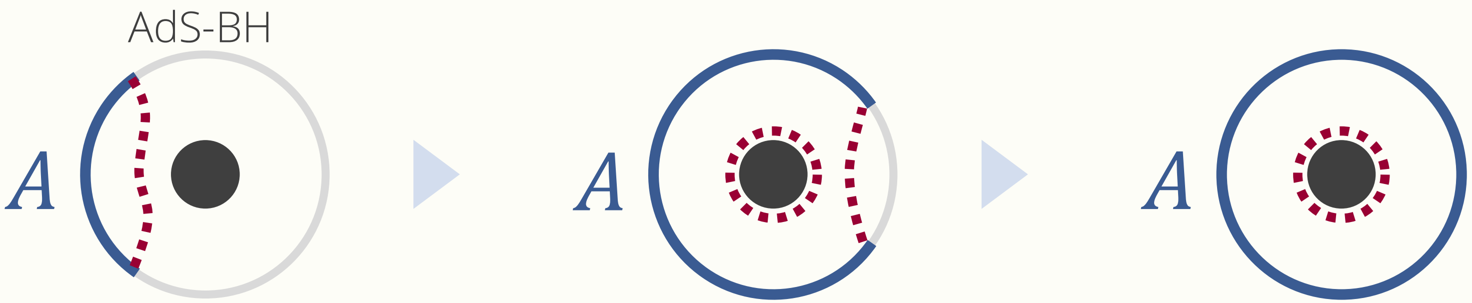


Time slice

$\gamma_A^{\min}$ Ryu-Takayanagi (RT) surface

Ryu-Takayanagi formula

Example AdS-black hole



Cf. homology condition

$$S_{CFT} = \frac{\mathrm{Area}(\gamma_{\mathrm{Horizon}})}{4G_N}$$

Ryu-Takayanagi formula

[Ryu-Takayanagi '06]

$$S_A = -\mathrm{Tr}\rho_A \log \rho_A = \min_{\gamma_A} \frac{\mathrm{Area}(\gamma_A)}{4G_N}$$

Entanglement entropy (CFT) Area of minimal surface (AdS)

Data of bulk geometry is encoded as quantum entanglement in the boundary field theory [Raamsdonk '09, '10]...

Built up spacetime from entanglement

Motivation

Ryu-Takayanagi formula is **not enough** to characterize the structure of entanglement in holography

Why?

Entanglement entropy can not measure quantum entanglement for mixed states

$$\rho_{AB} = |\Psi\rangle\langle\Psi|_{AB} \quad \blacktriangleright \quad \rho_{AB} = \Sigma_n p_n |\Psi_n\rangle\langle\Psi_n|_{AB} \quad (p_n \geq 0,\ \Sigma_n p_n = 1)$$

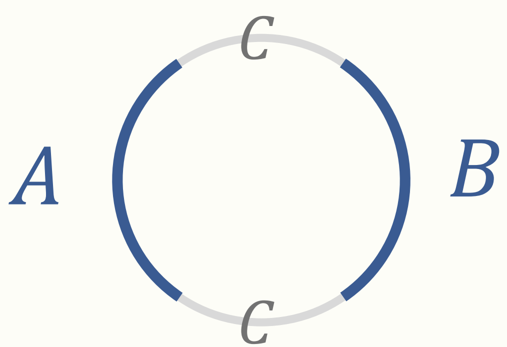
Pure — Mixed

E.g. S_A may be positive even for uncorrelated state $\rho_A \otimes \rho_B$ if ρ_A is mixed

Entanglement theory for mixed states is excellently formulated in terms of LOCC
(will not go into details) E.g. [Horodecki-Horodecki-Horodecki-Horodecki '09]

Motivation

A typical situation



ρ_{AB} is typically a mixed state if $B \neq A^c$

$$\rho_{AB} = \mathrm{Tr}_C[|\Omega\rangle\langle\Omega|]$$

Any of (S_A, S_B, S_{AB}) does not measure entanglement between A and B

Motivation

Our goal

Find a connection between geometry and entanglement in general situations (for mixed states)

Plan of talk

1 Introduction

AdS/CFT correspondence and Ryu-Takayanagi formula

2 Holographic Entanglement of Purification

3 Recent developments

4 Summary

Holographic Entanglement of Purification

Holographic EoP conjecture

Takayanagi-KU, Nat. Phys. 14 (2018) 6, 573-577
&
[Nguyen-Devakul-Halbasch-Zaletel-Swingle ’17]

$$E_P(\rho_{AB}) := \min_{|\psi\rangle_{AA'BB'}} S_{AA'} = \min_{\Sigma_{AB}} \frac{\mathrm{Area}(\Sigma_{AB})}{4G_N}$$

Entanglement of purification (CFT) Entanglement wedge cross section (AdS)

Entanglement of Purification

Purification

A purification of ρ_X

$$|\psi\rangle_{XX'} \text{ in } \mathcal{H}_{XX'} \text{ s. t. } \mathrm{Tr}_{X'}[|\psi\rangle\langle\psi|_{XX'}] = \rho_X \text{ on } \mathcal{H}_X$$

Example

$$\rho_X = \sum_i \lambda_i |i\rangle\langle i|_X$$ diagonalized

$$|\psi\rangle_{XX'} = \sum_i \sqrt{\lambda_i} |i\rangle_X |i\rangle_{X'}$$ X' is a "copy" of X

Purification is not unique
up to local unitary transformations $I_X \otimes U_{X'}$
and the dimension of auxiliary system $\mathcal{H}_{X'}$ ($\geq \mathrm{rank}\rho_X$)

Entanglement of Purification

[Terhal-Horodecki-Leung-DiVincenzo '02]

ρ_{AB} ▸ $|\psi\rangle_{AA'BB'}$

$$E_P(\rho_{AB}) := \min_{|\psi\rangle_{AA'BB'}} S(\rho_{AA'}) \qquad (\rho_{AA'} := \mathrm{Tr}_{BB'}[|\psi\rangle\langle\psi|_{AA'BB'}])$$

- Returns to entanglement entropy for $\rho_{AB} = |\psi\rangle\langle\psi|_{AB}$

$$E_P(|\psi\rangle\langle\psi|_{AB}) = S_A = S_B$$

- Quantifies entanglement and classical correlations between A and B as minimal entanglement between AA' and BB'

  E.g. $E_P(\rho_{AB}) > 0$ for separable states $\rho_{AB} = \sum_i p_i\, \rho_A^i \otimes \rho_B^i$

- An operational interpretation in terms of EPR pairs

$$\rho_{AB} \Leftarrow |\mathrm{EPR}\rangle_{AB}^{\otimes E_P(\rho_{AB})}$$

Local Operations and asymptotically negligible communication

Entanglement of Purification

- Satisfies the following information-theoretic properties [Bagchi-Pati '15]

(1) Faithfulness: $E_P(\rho_{AB}) = 0$ if and only if $\rho_{AB} = \rho_A \otimes \rho_B$

(2) $E_P(\rho_{AB}) \leq \min\{S_A, S_B\}$

(3) Monotonicity under partial trace: $E_P(\rho_{AB_1B_2}) \geq E_P(\rho_{AB_1})$

(4) $E_P(\rho_{AB}) \geq \frac{I(A:B)}{2}$

(5) $E_P(\rho_{AB_1B_2}) \geq \frac{I(A:B_1)}{2} + \frac{I(A:B_2)}{2}$

(6) Additivity (under a certain condition):

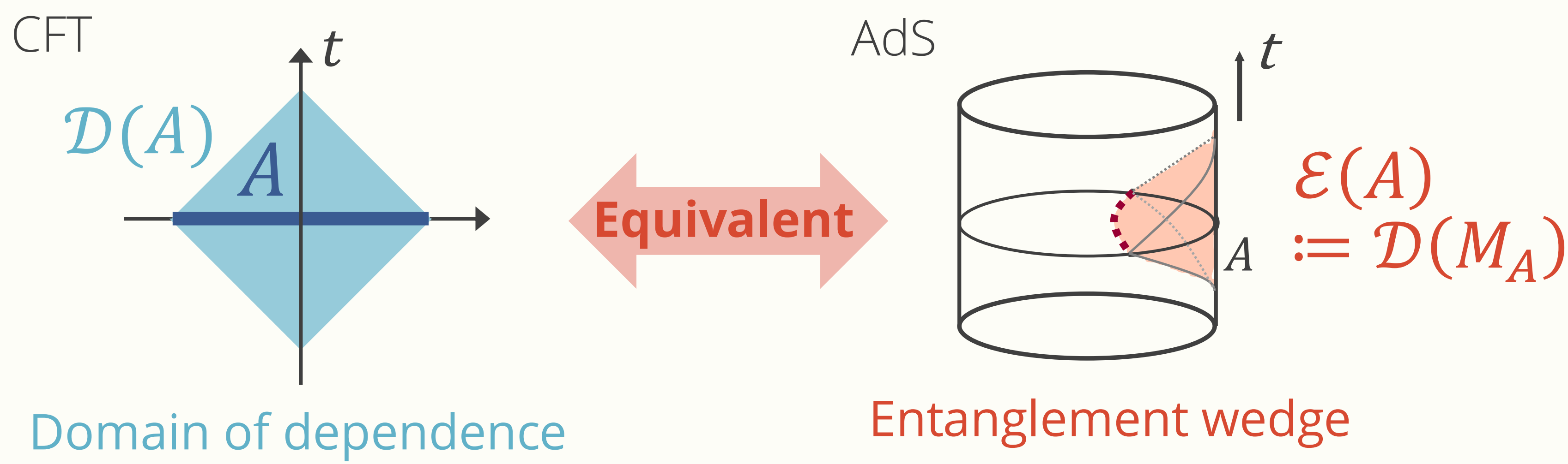
$$E_P(\rho_{A_1B_1} \otimes \rho_{A_2B_2}) = E_P(\rho_{A_1B_1}) + E_P(\rho_{A_2B_2})$$

Etc.

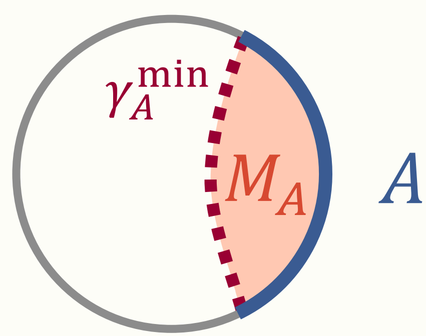
Entanglement Wedge Cross Section

Subregion/subregion duality

[Czech-Karczmarek-Nogueira-Raamsdonk ’12] [Wall ’12]
[Headrick-Hubeny-Lawrence-Rangamani ’14] …

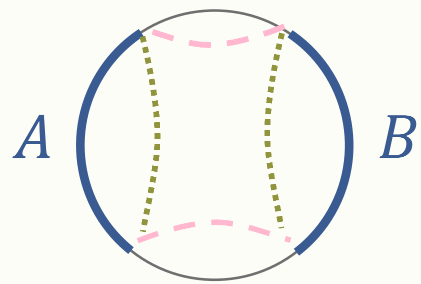


On a time slice



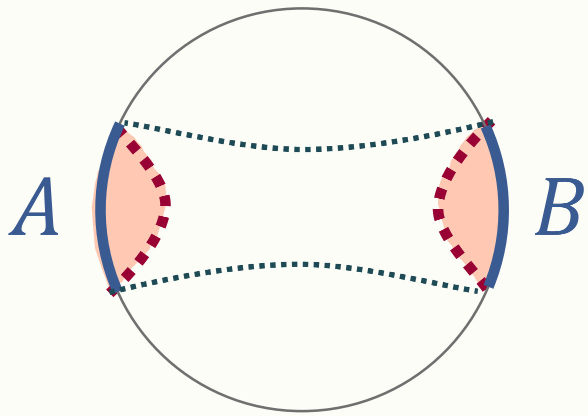
Entanglement Wedge Cross Section

Entanglement wedge of $A \cup B$



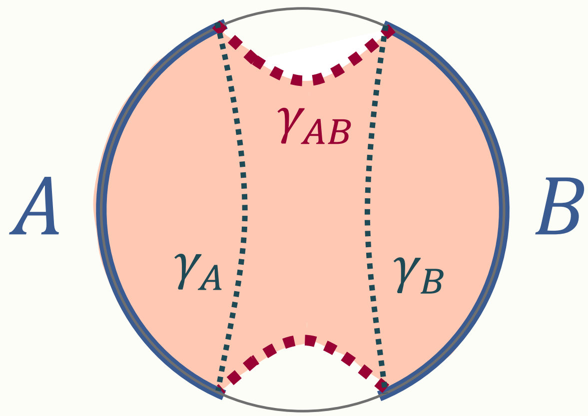
A and B are **distant**

$$M_{AB} = M_A \cup M_B$$



A and B are **close**

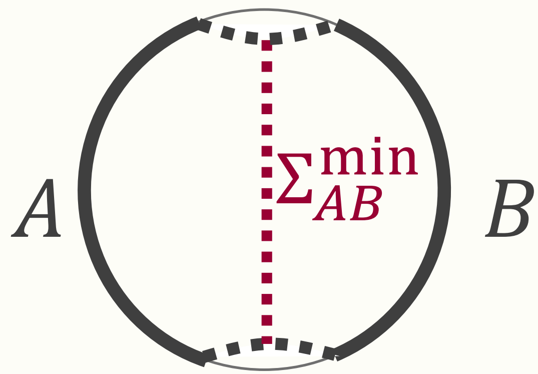
$$M_{AB} \neq M_A \cup M_B$$



Entanglement Wedge Cross Section

[Takayanagi-KU '17] [Nguyen-Devakul-Halbasch-Zaletel-Swingle '17]

$$E_W(\rho_{AB}) \coloneqq \min_{\Sigma_{AB}} \frac{\mathrm{Area}(\Sigma_{AB})}{4G_N}$$

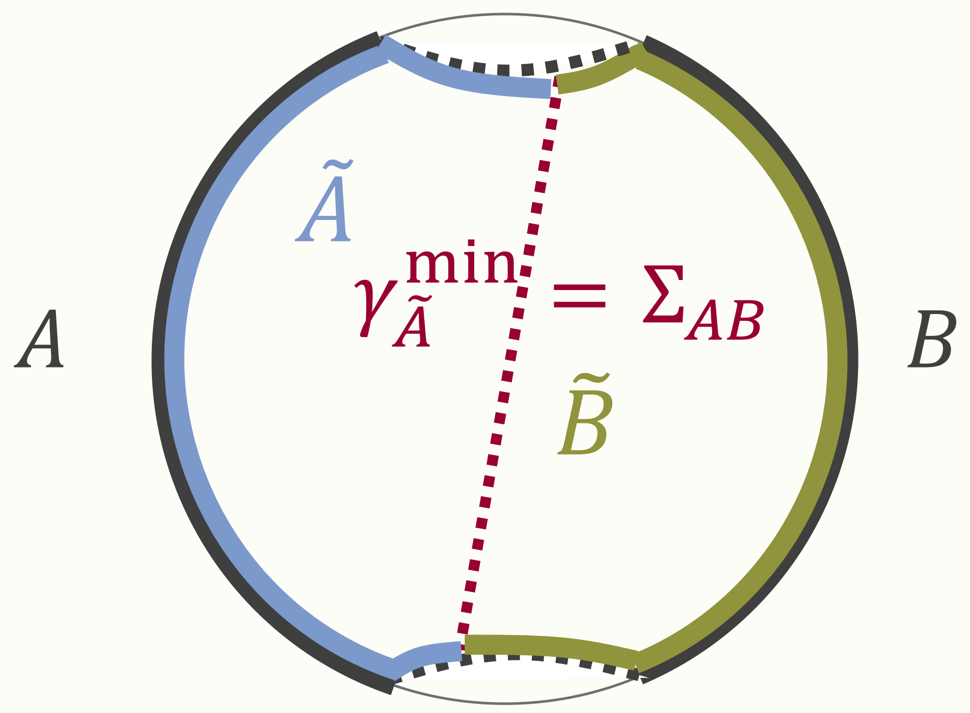


Formal definition

1. Divide the boundary of entanglement wedge ∂M_{AB} into $\tilde{A}$ and $\tilde{B}$ so that $\tilde{A} \supset A, \tilde{B} \supset B$
2. Compute $S_{\tilde{A}} = \min_{\gamma_{\tilde{A}}} \frac{\mathrm{Area}(\gamma_{\tilde{A}})}{4G_N} =: \frac{\mathrm{Area}(\Sigma_{AB})}{4G_N}$ for this $\tilde{A}, \tilde{B}$
3. Minimize $S_{\tilde{A}}$ over all possible divisions $\partial M_{AB} = \tilde{A} \cup \tilde{B}$

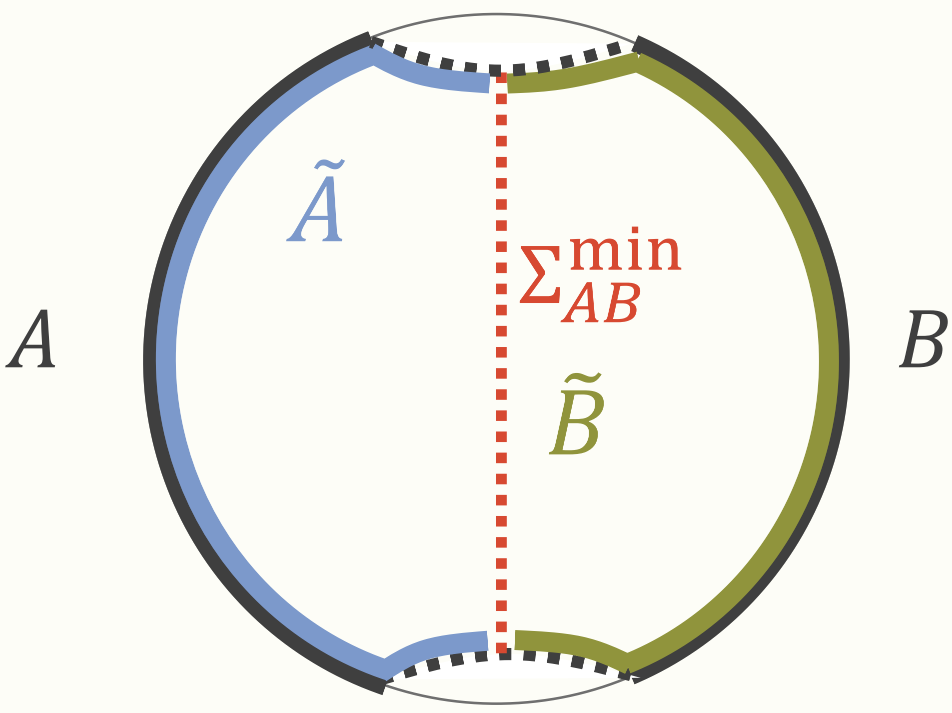
Entanglement Wedge Cross Section

Example1 Pure AdS



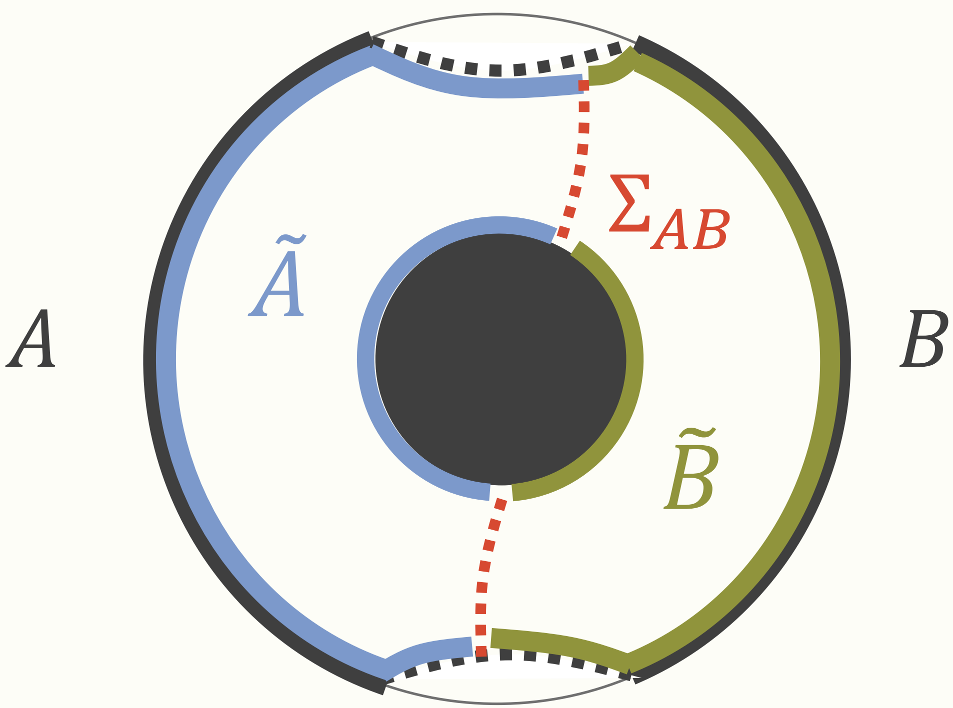
Entanglement Wedge Cross Section

Example1 Pure AdS



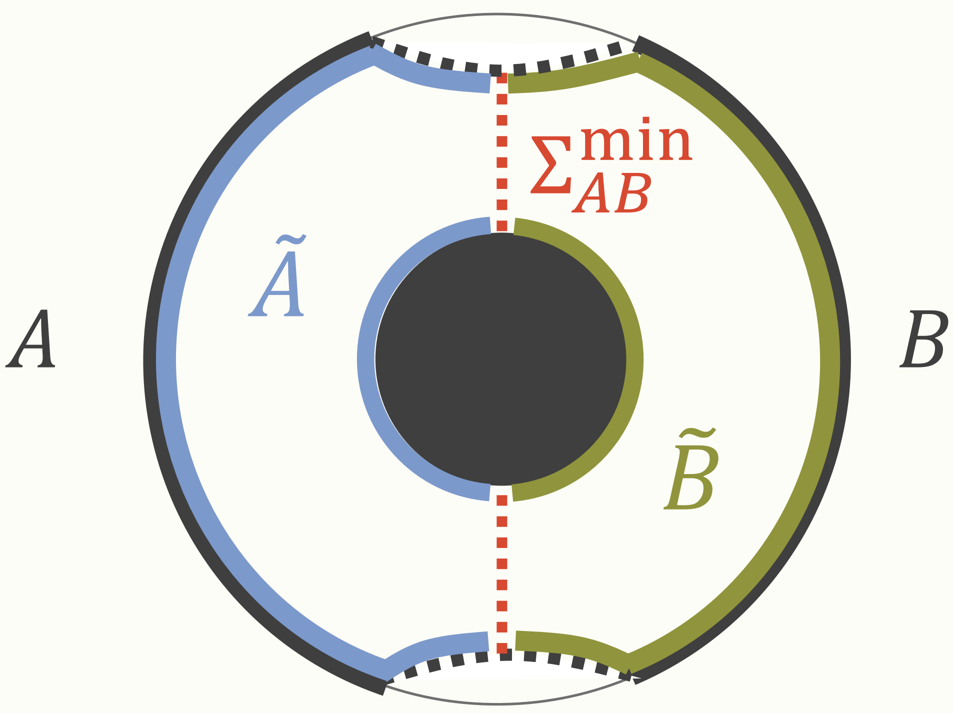
Entanglement Wedge Cross Section

Example2 AdS-black hole



Entanglement Wedge Cross Section

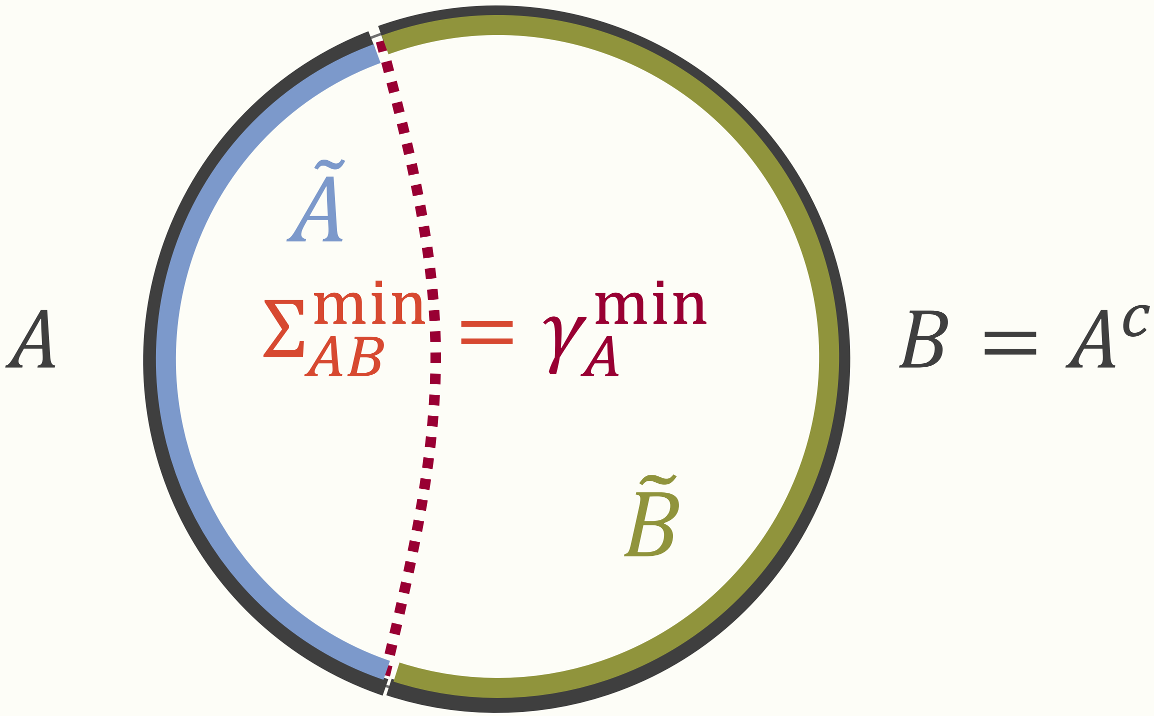
Example2 AdS-black hole



Entanglement Wedge Cross Section

EWCS is a generalization of Ryu-Takayangi surface

$E_W(\rho_{AB}) = S_A = S_B$ if ρ_{AB} is pure



Entanglement Wedge Cross Section

Moreover, EWCS satisfies all of known properties of EoP

(1) Faithfulness: $E_W(\rho_{AB}) = 0$ if and only if $\rho_{AB} = \rho_A \otimes \rho_B$

(2) $E_W(\rho_{AB}) \leq \min\{S_A, S_B\}$

(3) Monotonicity under partial trace: $E_W(\rho_{AB_1B_2}) \geq E_W(\rho_{AB_1})$

(4) $E_W(\rho_{AB}) \geq \frac{I(A:B)}{2}$

(5) $E_W(\rho_{AB_1B_2}) \geq \frac{I(A:B_1)}{2} + \frac{I(A:B_2)}{2}$

(6) Additivity (under a certain condition):

$$E_W(\rho_{A_1B_1} \otimes \rho_{A_2B_2}) = E_W(\rho_{A_1B_1}) + E_W(\rho_{A_2B_2})$$

Etc.

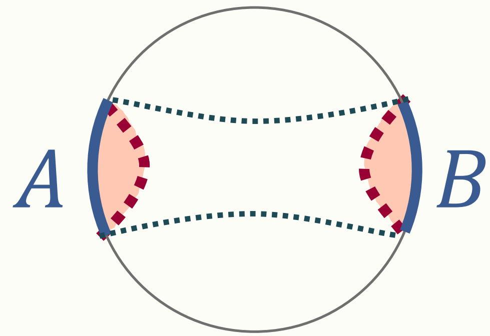
Entanglement Wedge Cross Section

Example1

$$E_W(\rho_{AB}) = 0 \Leftrightarrow \rho_{AB} = \rho_A \otimes \rho_B$$

Geometrical proof

$$E_W(\rho_{AB}) = 0 \quad \Leftrightarrow$$



$$\Leftrightarrow \quad \gamma_{AB}^{\min} = \gamma_A^{\min} \cup \gamma_B^{\min}$$

$$\Leftrightarrow \quad S_{AB} = S_A + S_B$$

$$\Leftrightarrow \quad \rho_{AB} = \rho_A \otimes \rho_B$$

Entanglement Wedge Cross Section

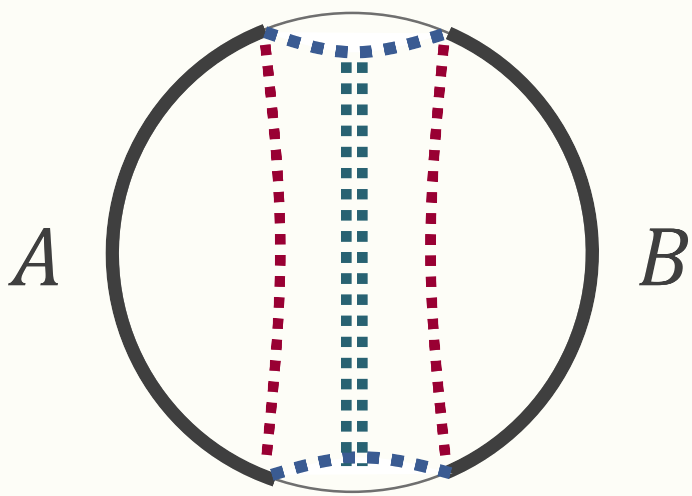
Example2

$$E_W(\rho_{AB}) \geq \frac{I(A:B)}{2}$$

Mutual information

$$I(A:B) := S_A + S_B - S_{AB}$$

Another measure of entanglement and classical correlations



Geometrical proof

$$2E_W(\rho_{AB}) + S_{AB} \geq S_A + S_B$$

$$\Updownarrow$$

$$E_W(\rho_{AB}) \geq I(A:B)/2$$

Holographic EoP conjecture

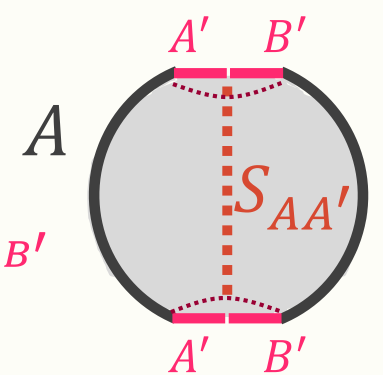
$$E_P(\rho_{AB}) = E_W(\rho_{AB})$$

Entanglement of purification (CFT) Entanglement wedge cross section (AdS)

- Supported by agreement of their various properties
- Interpretation from Tensor Network description of AdS/CFT

[Swingle '09] [Miyaji-Takayanagi '15] ..

$$E_P(\rho_{AB}) = \min_{|\psi\rangle_{AA'BB'}}$$



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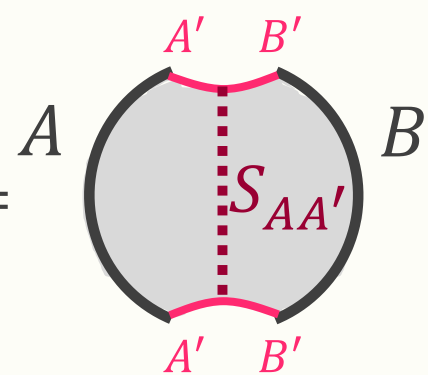

$$= E_W\,(\rho_{AB})$$

Assuming that the minimum is obtained by "geometrical states"

Plan of talk

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AdS/CFT correspondence and Ryu-Takayanagi formula

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Recent developments

Recent developments

There have been lots of progress in many directions

A part of such works

- Bit thread formalism

  [Freedman-Headrick ’16]
  [Agón-Boer-Pedraza ’18] [Du-Chen-Shu ’19]...

- EWCS by other information measures

  [Kudler-Flam-Ryu ’18]
  [Tamaoka ’18]...

- Complexity of purification

  [Cáceres-Couch-Eccles-Fischler ’18]
  [Ghodrati-Kuang-Wang-Zhang-Zhou ’19]...

- EWCS for Higher dimensional black holes/Lifshitz holography

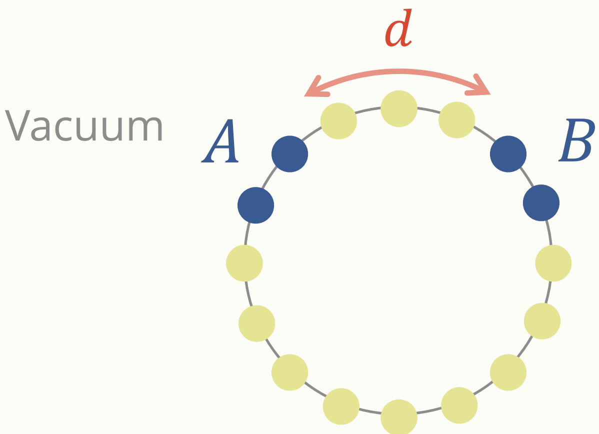
  [Yang-Zhang-Li ’18]
  [Velni-Mozaffar-Vahidinia ’19]...

EoP in Many Body Systems

Bhattacharyya-Jahn-Takayanagi-KU, To appear in PRL

Numerical calculations of EoP

(1) 2d free scaler field theories on a lattice (2) 2d transverse-field Ising model



N: total cites number
w: width of subsystems $|A| = |B|$
d: # of cites between A and B

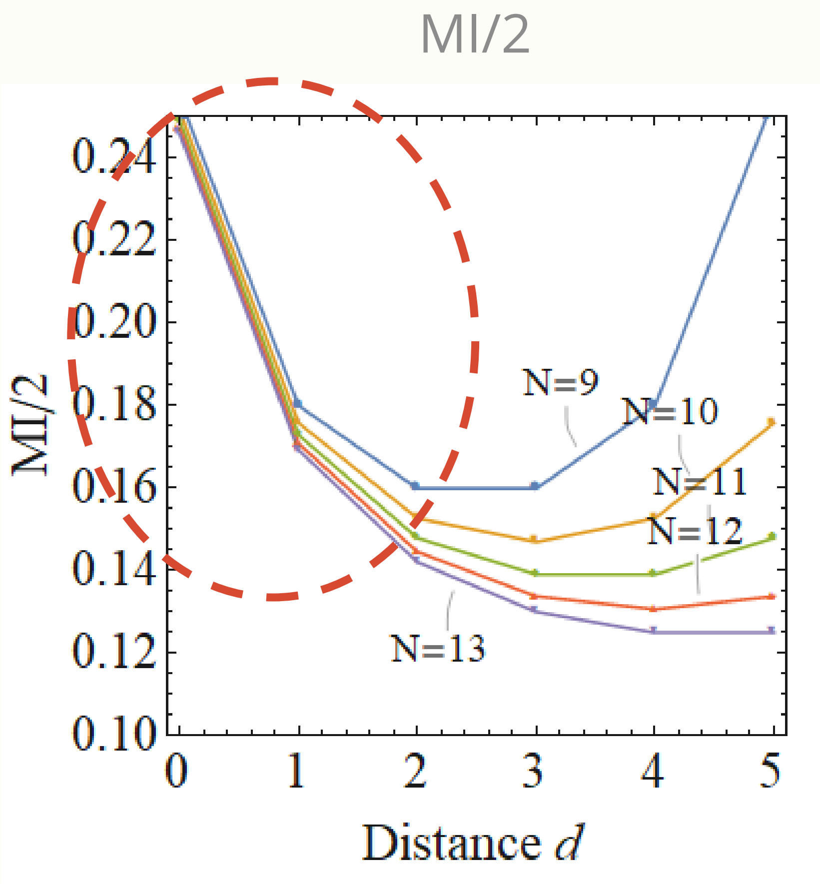
The minimization of EoP is implemented by

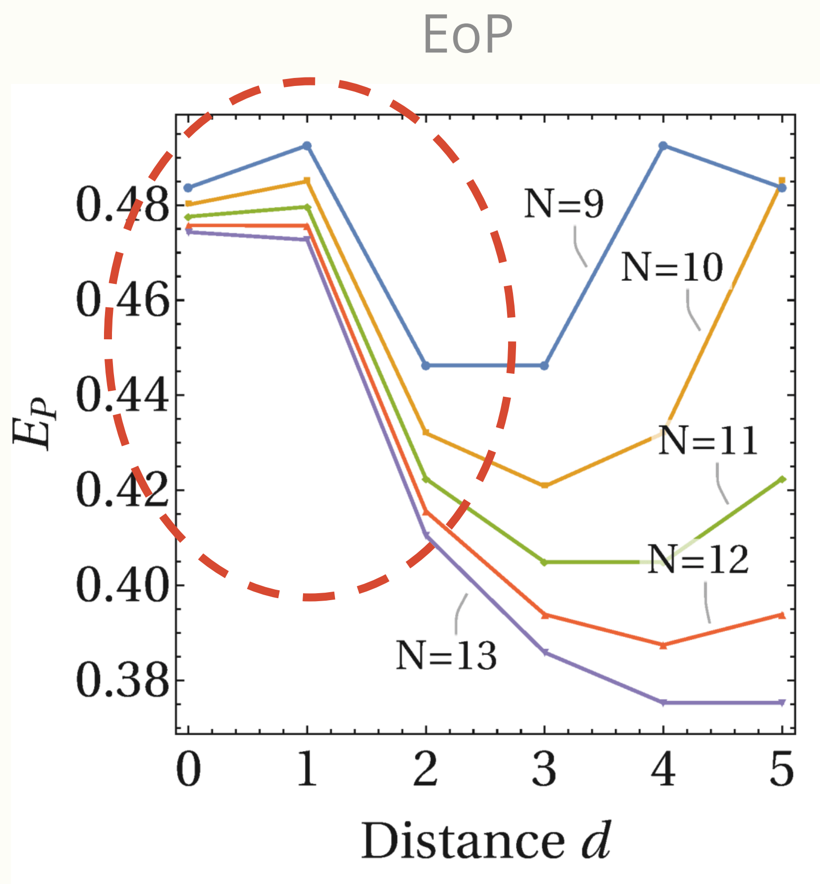
(1) - minimal Gaussian purification ansatz $\Psi^{\mathrm{G}}_{AA'BB'}(\vec{\phi}) = \mathcal{N}\exp\left(-\frac{1}{2}\vec{\phi}^T V \vec{\phi}\right)$

(2) - Full minimization over all possible purifications of the dimensions $\{\mathrm{dim}\mathcal{H}_{A'}, \mathrm{dim}\mathcal{H}_{B'}\} \leq \mathrm{rank}\rho_{AB}$ [Ibinson-Linden-Winter '06]

EoP in Many Body Systems

(I) EoP becomes a non-monotonic function of the distance d for small N



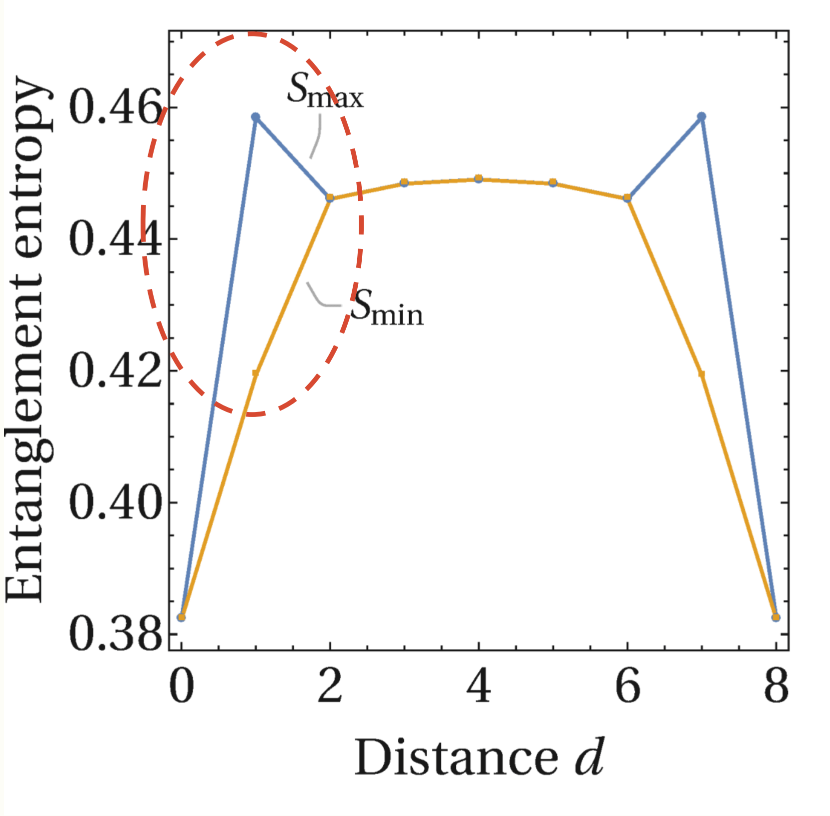


(critical Ising, $w = 2$)

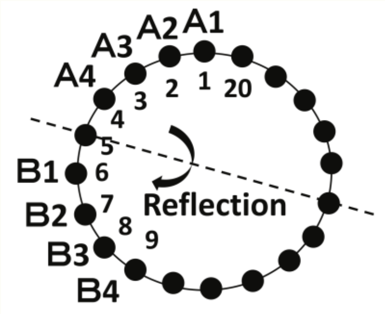
EoP in Many Body Systems

(II) The symmetry exchanging A and B breaks in optimal purifications

$$S_{\max} = \max\{S_{A'}, S_{B'}\}$$
$$S_{\min} = \min\{S_{A'}, S_{B'}\}$$



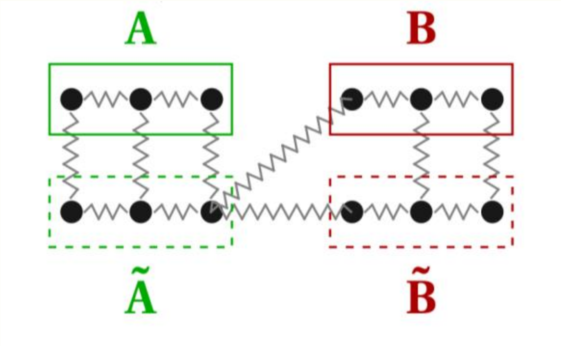
(critical Ising, $w = 1$)



Reflection symmetry $\rho_{AB} = \rho_{BA}$

Can be non-trivially broken in optimal purifications

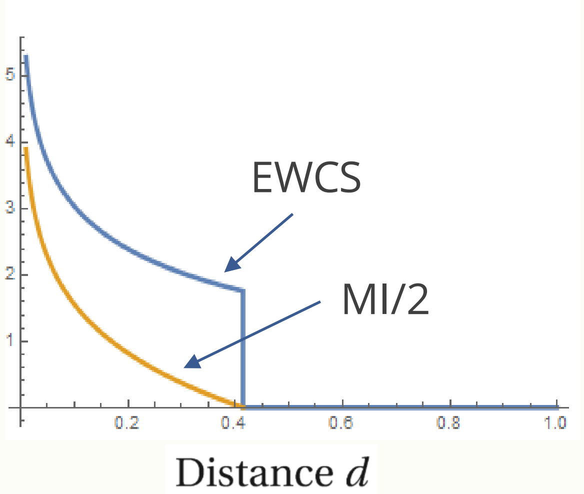
$$S(\rho^*_{A'}) \neq S(\rho^*_{B'})$$



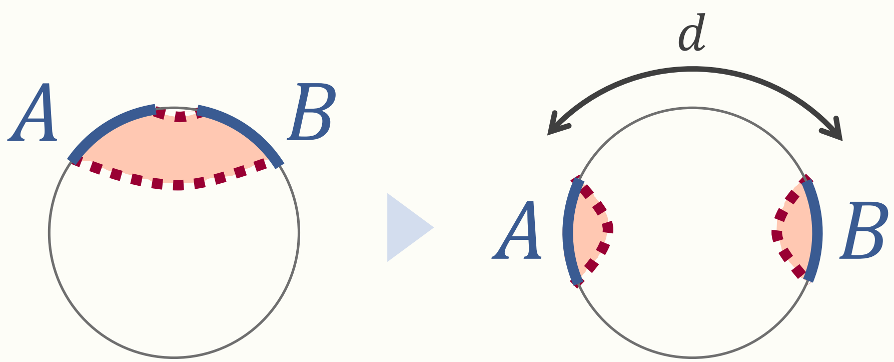
EoP in Many Body Systems

- Both results (I), (II) can be explained as an interplay between quantum entanglement and classical correlations

- A potential relation to holographic EoP conjecture



Holographic calculation of MI and EWCS



(I) EWCS behaves differently than MI around the transition point of M_{AB}

(II) Reflection symmetry could be broken in excited states or $O(1)$ correction

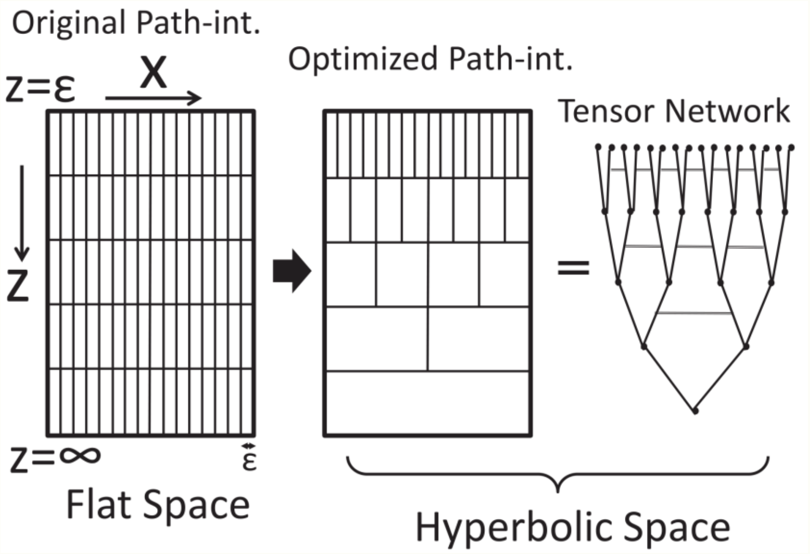
HEoP from Path integral complexity

Caputa-Miyaji-Takayanagi-KU, Phys. Rev. Lett. 122 (2019) no.11, 111601

Path integral complexity

Path integral optimized by "minimal complexity" criteria in CFTs

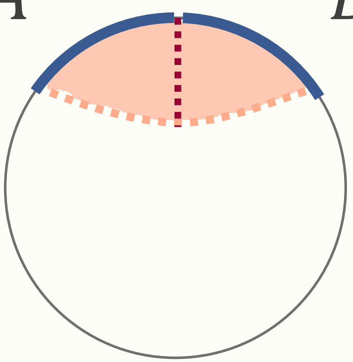
[Caputa-Kundu-Miyaji-Takayanagi-Watanabe '17]...



It forms a subclass of purifications of ρ_{AB}
where $|\psi^{\mathrm{PI-opt.}}\rangle_{AA'BB'}$ has the minimal complexity

HEoP from Path integral optimization

The $S(\rho_{AA'}^{\mathrm{PI-opt.}})$ over the minimal complexity purifications exactly agrees with EWCS when A and B are adjacent ($d \to 0$)



$$E_W(\rho_{AB}) = \min_{A'} S(\rho_{AA'}^{\mathrm{PI-opt.}})$$
$$= \frac{c}{6}\log[1 + 2z + 2\sqrt{z(z+1)}]$$

$A = [a_1, a_2],\ B = [b_1, b_2]$ $\quad z \equiv \frac{(a_2-a_1)(b_2-b_1)}{(b_2-a_1)(b_1-a_2)}$

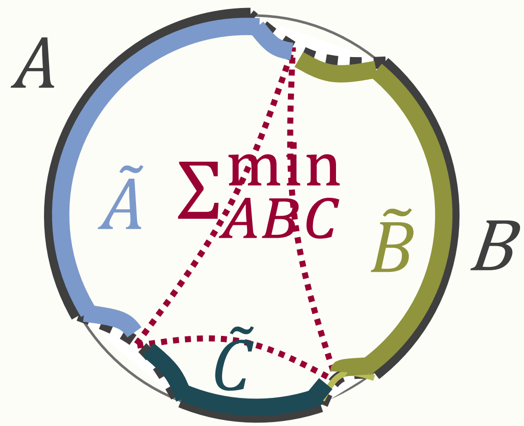
Minimal complexity ≅ Minimal entanglement (?)

$S(\rho_{AA'}^{\mathrm{PI-opt.}})$ slightly deviates from EWCS for $d > 0$ where PI optimization in untrustable

Multipartite holographic EoP conjecture

KU-Zhou, JHEP 1810 (2018) 152

A generalization of EoP/EWCS for multipartite systems



$$E_P(\rho_{A_1\ldots A_n}) := \frac{1}{2} \min_{|\psi\rangle_{A_1A_1'\ldots A_nA_n'}} \left[\Sigma_{i=1}^{n} S_{A_iA_i'}\right]$$

$$E_W(\rho_{A_1\ldots A_n}) := \frac{1}{2} \min_{\Sigma_{A_1\ldots A_n}} \frac{\text{Area}(\Sigma_{A_1\ldots A_n})}{4G_N}$$

Multipartite holographic EoP conjecture

We confirmed the perfect agreement of their various properties

(1) $E_{P,W}\left(|\psi\rangle_{A_1 \ldots A_n}\right) = \sum_i S_{A_i}/2$
(2) $E_{P,W}\left(\rho_{A_1 \ldots A_n}\right) = 0$ if and only if $\rho_{A_1 \ldots A_n} = \bigotimes_{i=1}^{n} \rho_{A_i}$
(3) $E_{P,W}\left(\rho_{A_1 \ldots A_n}\right) \leq \min_i [S_{A_1} + \cdots + S_{A_1 \cdots A_{i-1} A_{i+1} \cdots A_n} + \cdots + S_{A_n}]/2$
(4) $E_{P,W}\left(\rho_{A_1 \ldots A_{n-1}(A_n \cup A_n')}\right) \geq E_{P,W}\left(\rho_{A_1 \ldots A_{n-1} A_n}\right)$
(5) $E_{P,W}\left(\rho_{A_1 \ldots A_{n-1}(A_n \cup A_n')}\right) \geq \frac{T(A_1:\cdots:A_n)}{2}, \frac{D(A_1:\cdots:A_n)}{2}$
(6) $E_{P,W}(\rho_{ABC}) \geq (E_{P,W}(\rho_{AB}) + E_{P,W}(\rho_{BC}) + E_{P,W}(\rho_{CA}))/2$
Etc.

The holographic EoP conjecture is strengthened and generalized to

$$E_P\left(\rho_{A_1 \ldots A_n}\right) = E_W\left(\rho_{A_1 \ldots A_n}\right)$$

Summary

We proposed the holographic entanglement of purification conjecture as a generalization of Ryu-Takayanagi formula

$$E_P(\rho_{AB}) := \min_{|\psi\rangle_{AA'BB'}} S_{AA'} = \min_{\Sigma_{AB}} \frac{\text{Area}(\Sigma_{AB})}{4G_N}$$

- It is supported by various consistency checks and interpretation
- We calculated EoP in free scalar and Ising model by numerics and found non-monotonicity and reflection symmetry breaking
- We calculated EoP within minimal complexity purifications in CFTs and found the agreement with EWCS for adjacent $A \cup B$
- We generalized holographic EoP conjecture to multipartite systems

Future directions

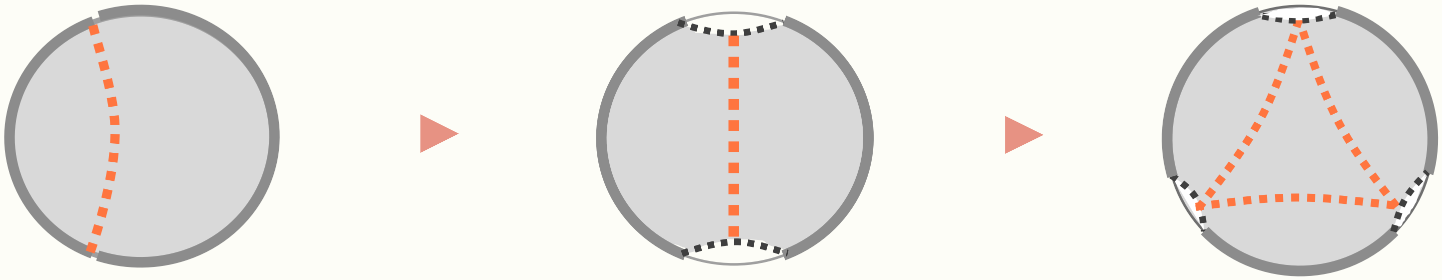
- Analytical calculation of EoP in holographic CFTs

- EoP in strongly interacted many body systems

- EWCS and reflection symmetry breaking

- Applications
  - Kinematic space
  - Constraints on holographic states
  - Squashed entanglement

- An operational interpretation of multipartite EoP



Thank you for your attention